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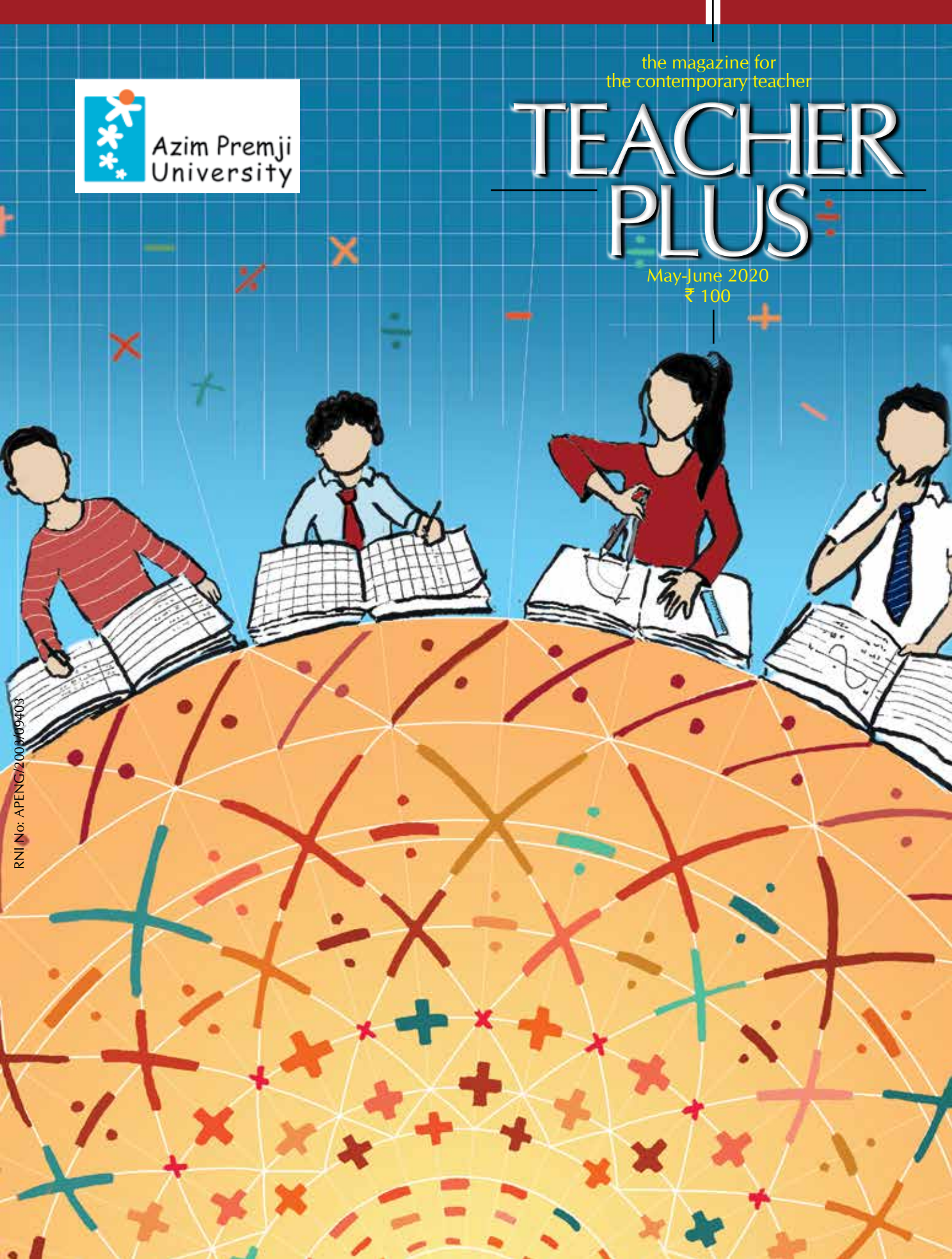

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DEVELOPING LEADERS for SOCIAL CHANGE

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EDITORIAL

Amid the uncertainty...

Just like much of the rest of the world, the *Teacher Plus* team has been working from home, sheltering in place, and trying to keep up with what needs to be done. It's been a strange couple of months, knowing that while many of us are fairly comfortable both in terms of having a place to stay and the access to Internet that allows us to remain connected and productive, there are millions of others who have neither. We've all become unfortunately familiar with the images of people fleeing cities in any way they could, hoping to find some succor in their home villages. In almost all the pictures, there are children – hanging on to their parents, clutching at a mother's hand, sitting by the side of the road, tired and hungry. These have been the most heartbreaking, and reflect the most urgent aspect of the crisis. But there are many other issues on the slow boil that will perhaps emerge in the not too distant future, across many fields of human activity.

In education, things have come to a virtual (no pun intended) standstill. Apart from a handful of elite institutions whose learning and teaching communities have good internet access and the appropriate devices, the vast majority, from primary to tertiary levels, have struggled with online classes and most have given up. Suddenly the promising rhetoric of a technology-driven system has collapsed, with the gaps in preparedness, infrastructure, and readiness only too visible. It's been realized – keenly – that context and sociality play a huge role in learning, and this is particularly true for young children. While those who have home schooled for years have the advantage of experience here, others, more used to the regimens and the physical structures of conventional education have been at a loss. Even without the huge gaps in access, it has been brought home to us that there are many other variables that we need to consider when we move teaching/learning online.

The other major issue is stress and anxiety stemming from the general climate of uncertainty. How does one educate or prepare young people for a future whose shape is so uncertain? What are the ways in which the social and economic world will change, and how do we equip ourselves for these changes? What are the new habits of living that we will have to acquire and quickly put into practice? We're all fumbling for answers, looking to science and social policy and public health experts to give us some guidance about how to plan.

But in the middle of all that, we hold on to some things. This special issue represents one of those attempts – to focus on the beauty and steadfastness of mathematics in all its infinite variations. This is the second time we have focused on mathematics teaching, and it won't be the last. Our guest editor, Sundaram S, has done a remarkable job of pulling together a diverse set of articles that push us to question and rethink how we have dealt with the subject, and give us some inventive ways to bring the magic of math alive in the classroom – or perhaps, on a screen!

Stay safe. And keep learning.

Usha Raman

Editor
Usha Raman

Editorial Team
M Nirmala
Shalini B
Sushma Rana

Layout & Graphics
Rajendra Kumar S

Social Media
Jamuna Inamdar

Circulation
N Srinivas

RNI Publications Consultant
K Raghurama Raju

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Teacher Plus
A 15, Vikrampuri,
Secunderabad 500 009.
Telangana. India.
Tel: 040 2780 7039.
editorial@teacherplus.org
www.teacherplus.org

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Cover Illustration
Jisha Unnikrishnan is an illustrator, textile designer and researcher based in Ahmedabad. More of her work can be seen on her website www.jishaunnikrishnan.com. She can be reached at <jisha.mail@gmail.com>.

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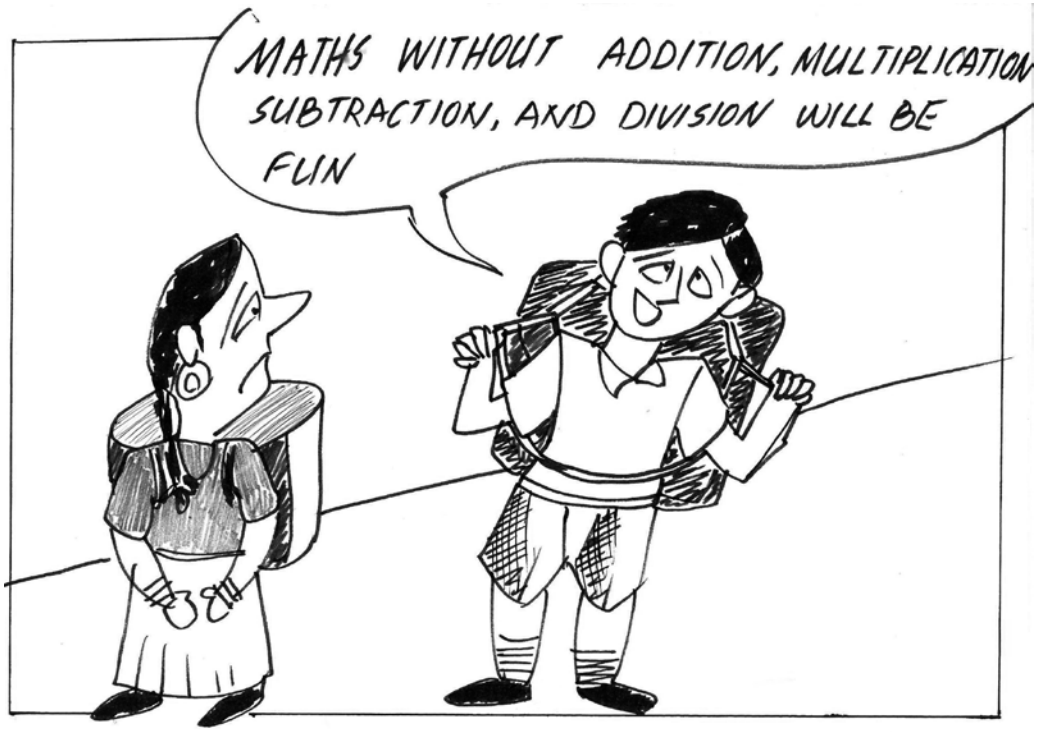
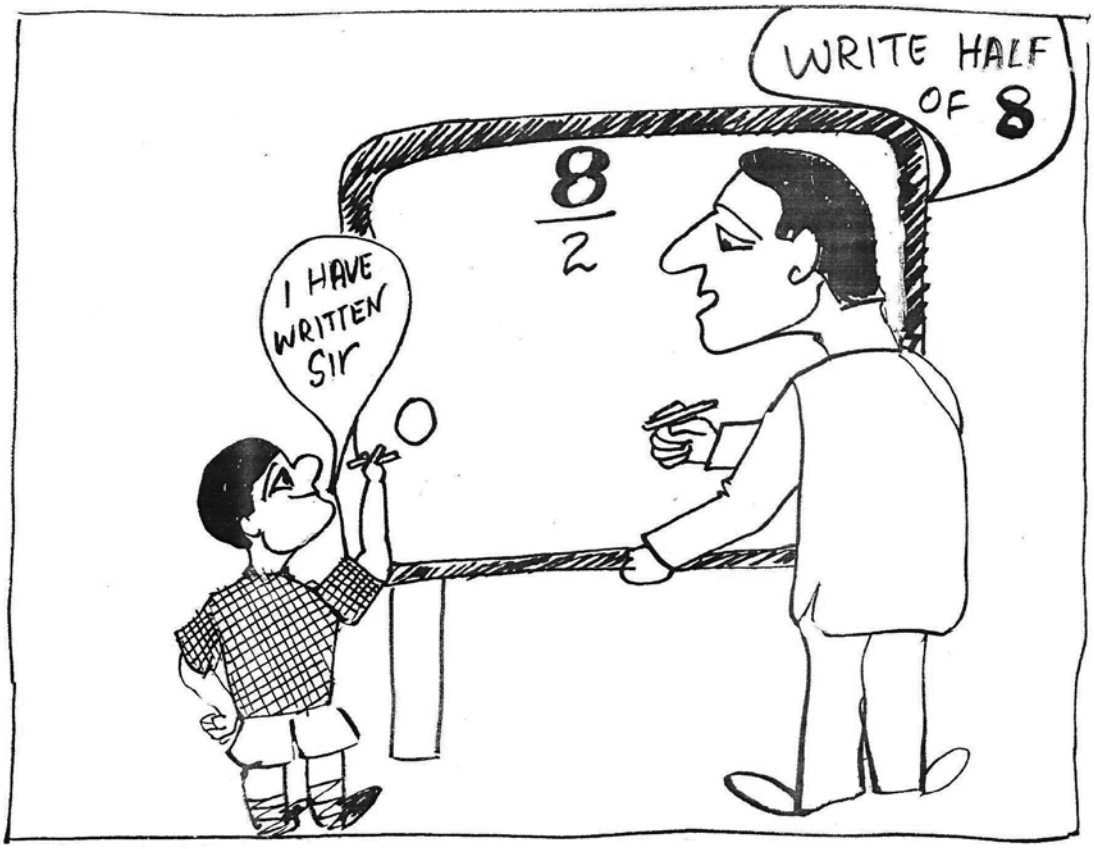




All cartoons in this issue by
VSS Sastry

Reinventing math

S Sundaram

It was in early January that *Teacher Plus* asked me if I would be interested in guest-editing a special issue on math to be published in June 2020. I said yes immediately almost without thinking. Only a few hours later did I realize the enormity of the task I had committed to!

But a discussion with a few close professional friends and a review of the previous math special issue (published in June 2009), enabled me to draft a road map, draw up a list of topics and possible writers. The main focus was to be on the improvement of the quality and enrichment of math instruction in our schools.

After reviewing all the articles we received, we decided to group them under three broad sections.

"Policy and Perspectives". In today's globalized world, our students compete with the rest of the world and one of the crucial determinants of their success will be their competency in math. There is a need to compare our curriculum and performance globally. It is an accepted fact that the quality of math teaching and learning in India needs a lot of improvement. The national education policy, covering among other things, curriculum, pedagogy, assessment and teacher education, is one of the major instruments that can lead to the much needed improvement. Therefore, there is a need to critique the policy and its implementation.

"Pedagogy" which explores various issues related to teaching and learning of math in our schools. It points out aspects of math which have not been fully exploited by the curriculum. Math has acquired a "not-so-deserved" reputation of being a boring and difficult subject. However, the study of math has also given rise to a lot of puzzles, games and pastimes. Math improves deep thinking and deep thinking is pleasurable. Math provides for edutainment! One article even questions if math has been given too much importance in our schools!

Finally we have "Practice". In spite of the bleak landscape at a pan-India level, a number of teachers, educators and schools are making efforts to provide students with rich learning experiences. Many of these efforts are also aimed towards students with disabilities of various kinds. Their motto, "think globally but act locally" needs wider acknowledgement and encouragement.

The range of the writers is varied – from teachers, principals, teacher educators and researchers to amateur enthusiasts! Age seems to be no bar to enjoying math, two of our oldest contributors being in their 80s! There is even distribution between men and women, contrary to a common perception that women avoid math. There is a delightful collection of math cartoons which appear throughout the issue. I hope readers will find the issue both entertaining and educative.

For me, it has been a wonderful opportunity, both for discovering new areas in math and getting to know youngsters who are also passionate about math. I would like to thank all the writers whose articles have made this issue a valuable resource. Thank you also to *Teacher Plus*.

Somewhat ironically, I have the Covid-19 lockdown to thank for giving me enough time to work on this project without too many distractions.

Finally thank you Gananth and Jayasree for your professional guidance and suggestions. Without your help it would have been very difficult for me to do justice to the project.

Exploring explorations

Vinusha Venkatesh

The author is a former fellow at Teach for India. She can be reached at <vinusha.v2017@teachforindia.org>.

Satish and Nitin are packing up after school. Today they had worked on solving triangular arithmagons. Nitin is happy; he had made progress in the hour, coming up with the equation '2 x (sum of the vertex numbers) = sum of the numbers assigned to the sides'. He'd been asked to check this for quadrilateral arithmagons.*
Satish eggs him on, saying, "Nee oru formula-veh kandu pudikkalaam da!"
"You can create your own formula, man!"

Mathematical explorations are mathematical activities around open-ended and loosely-defined problems that involve students asking their own questions, choosing the ones that interest them, following different paths to find answers and asking more questions. Key to the experience is the freedom to explore different approaches and students gaining a sense of ownership in creating mathematics.

For the past seven months, I had been observing Satish, Nitin and other 9th graders from two State Board schools navigate through *mathematical explorations* for an hour or two every week, as part of a research study. My primary task was to observe how students approached the problem posed; what they said, or did and note that down in an unbiased way, akin to a birdwatcher observing species in their natural habitat. However, what started as merely a documentation exercise, turned out to reveal fascinating things about student thinking and behaviour through these sessions. Here's capturing some of those interesting realizations and what they mean to me.

Math doesn't have to be scary

Class begins after regular school hours and as we step in, Prashant throws in a jovial, *"Inikki enna panna poroam ma'am?"*, loosely translating to, 'What are we doing today, ma'am?' 'Let's do something new!' He shows no signs of fear or boredom. The class starts with a simple question, something all students can easily access. 'Think of your age, then add 2, multiply by 4, divide by 2, subtract 3. What do you get?' Prashant shares his final answer and we guess his age correctly. Curiosity sparks!

Students are now creating their own 'magic number puzzles'. Everyone is working at their own pace and spending the time to figure out how to guess their friend's age. A follow-up question enters the mix to nudge them out of their comfort zone: 'How many different puzzles can you create where the answer is exactly 5, 21, 79, or any other given number?' Students are now playing around with the combinations. All variations are welcome.

'Now, modify the puzzle so that the answer is the number that you thought of.' As the class thinks of ways to return to the same number, Nisha gives it a name: *'loop'*. It sticks, and soon there are a flurry of loops.

Six months later, when you'd think that students would have forgotten this exercise, Nisha confronts us with an example of a loop she came across elsewhere. Watching her eyes light up as she made the connection to a class that happened quite a while ago, I think of how when students are given the freedom to work things out in their own way, there is joy and ownership. And when they do eventually discover or create something, it stays with them for a long time.

Think like a mathematician

Today, a clapping game is underway. Fifteen students sit in a circle and clap as instructed: every 3rd person claps, every 4th person and so on. They notice that if the student who starts the game, claps a second time before all of them have clapped, then the cycle repeats with only a select few clapping. They see this happen when every 3rd or 5th person claps; but when the 2nd or 4th person claps, all 15 students clap. Now they wonder, 'When exactly will *everyone* get to clap?'

Guided by the case where every 3rd and 5th person claps, Akshay makes a guess, "15 is odd, so if every odd numbered person claps, then all will not get to clap." By this logic not everyone should clap if every 7th person claps. They try it out and Akshay is proven wrong; everyone claps! More guesses emerge – '3 and 5 are factors of 15; so if every 3rd or 5th person claps, everyone won't clap'. This is sufficient to explain the anomaly in Akshay's guess. However, as students test for every 9th and 10th person, expecting everyone to clap, they find that they have to reject their second guess as well. Even though 9 and 10 are not factors of 15, only a few get to clap.

They now move from actual clapping to calling out numbers; if every 3rd person claps in a group of 18, then the 3rd, 6th, 9th, 12th, 15th and 18th person will repeatedly clap. The mathematization gives them the flexibility to try out more cases without physically clapping, and with more trials and guesses, they try to find a rule. We see them come up with a conjecture, test it out, discard it if false, revise and verify it again, till it works for all the examples. Perhaps due to a lack of fluency in mathematical language, they are unable to prove their conjecture or succinctly express the condition

Illustration: Sunil Chawdiker

when everyone would clap. But they definitely know the rule and how it works. They tackled an unfamiliar problem and worked like mathematicians to solve it.

'Multiply apple twice to get orange'

Not everything is rosy though, in this experience. We encountered some serious roadblocks to progress. Having usually solved problems where conditions are 'fixed' or given, students are habituated to look only for enough parameters to plug into a formula. This came out starkly when we asked students who were quite familiar with the set identity $\mathbf{n}(A \cup B) = n(A) + n(B) - n(A \cap B)$, for the total number of students in a class; where 20 students played football (A), 30 played cricket (B) and 25 played both games (A and B). Ranjith confidently answers, "There are totally 25 students in the class." No other student objects. Even though the answer (25) is procedurally valid, it does not occur to anyone that it's impossible for a class to have 25 students playing both cricket and football, when there are 20 students who play football alone.

One of the more troubling realizations we had was that students tended to interpret algebraic expressions in terms of values and operations (*'textbook talk'*) rather than its intended meaning. If **P** were number of apples and **Q** were number of oranges, what does $2P = Q$ mean? Shruti blurts out, "Multiply apple twice to get orange," which doesn't make much sense.

Despite the disconnect with the context of the problems, there were surprises too. For instance, in the fruit problem, the 'textbook talk' did not seem to extend to the equation where the idea of money was involved. If x was the price of an apple and y, the price of an orange, what would $Px + 15 = 50$ mean? Prashant responds instantly, "I bought apples for 35 rupees."

Now we've got questions to think about. Why was $Px + 15 = 50$ so easily interpreted? How did students connect the value 15 to money and not fruit quantity? We noticed that even a slight change in the context or framing of a problem, drastically changes student response.

For instance, in the clapping game, we had started by telling students to clap by 'skipping 2' persons..., which didn't help them to 'see' the number 3 as a factor of 15. In the subsequent runs, we amended our framing accordingly to 'every 3rd person claps... etc. Finding the appropriate context for students to connect to, and the right framing that gets the best out of them has been an ongoing challenge.

Reflections of an educator

Having worked with students in schools where lessons are time-bound, procedure heavy and the 'right' answer is valued more than the question; it's not hard to see how students often lose themselves in repetitive practice drills without fully understanding the purpose of the subject.

In contrast, explorations are inquiry based and student centric, where teachers facilitate and give inputs only when absolutely necessary. Doubts, misconceptions are often addressed by peers themselves. Student thinking is made visible. Varied mathematical approaches are encouraged and questions eagerly welcomed. There is immense scope for students to build abstraction, analytical thinking and problem solving capacities.

Each time I reflect on my observations, I notice how in an exploratory classroom, students' responses directly feed back into pedagogy, allowing one to alter their instructional course as needed. Unlike in a drill-practice centred lesson, kinks in lesson planning or execution are glaringly visible (think 'skipping 2 people'). Conceptual breakdowns and logic gaps in students are relatively clearer ('textbook talk') and therefore easy to remediate.

My hope is for more educators to take mathematical explorations to their classrooms and unleash the potential of a different, joyful and more authentic learning experience for students. It is without hesitation when I say that this has been a captivating and enlightening journey for me as well, for along with the students, I have become an explorer too!

**Arithmagons are puzzles where one has to find the unknown numbers at the vertices of a polygon, given the sum of the numbers along each side.*

Student names have been changed in the article.

TP

What does mathematics have to do with social justice?

Chintan Girish Modi

The author is an educator, writer and researcher who is deeply interested in social justice issues. He has an M.Phil. in English Language Education and several years of experience in facilitating workshops with students and teachers. He can be reached at <chintan.prajnya@gmail.com>.

Mathematics was a challenging subject for me in school. While some of my friends were able to solve mathematical problems in a jiffy, I used to struggle and often sought out the teacher's support. I managed to do well academically because of sheer persistence but never developed a love for mathematics. Perhaps the reason behind this was a lack of understanding about its place in my everyday life. It was something that just had to be done. I think my relationship with mathematics might have been different if I had known why it was on the school timetable.

When I read the Draft National Education Policy published by the Government of India's Ministry of Human Resource Development in 2019, I was struck by a recommendation that the mathematics curriculum should include "regular activities that explore connections between classroom mathematics and real-life mathematics." It was heartening to note the emphasis on getting children excited about mathematics and to channelize their excitement towards cultivating "logical skills that are so critical throughout their school years and indeed throughout life."

I wanted to know more about how this could be done and that is when I stumbled upon a booklet titled *A Guide for Integrating Issues of Social and Economic Justice into Mathematics Curriculum* authored by Jonathan Osler in 2007. It is free to download from radicalmath.org, "A resource for educators interested in integrating issues of social and economic justice into their math classes and curriculum." If you feel intimidated by American spelling and references, be patient and you will soon learn how to sift out the ideas that are relevant for your students.

Social justice mathematics is appealing to me because it answers a question that I have held in my mind for a long time: Why do I need to know this? It pulls me out of my own experience of mathematics as a subject that was all about memorizing and applying rules that did not make sense to me even if they made sense to the teacher

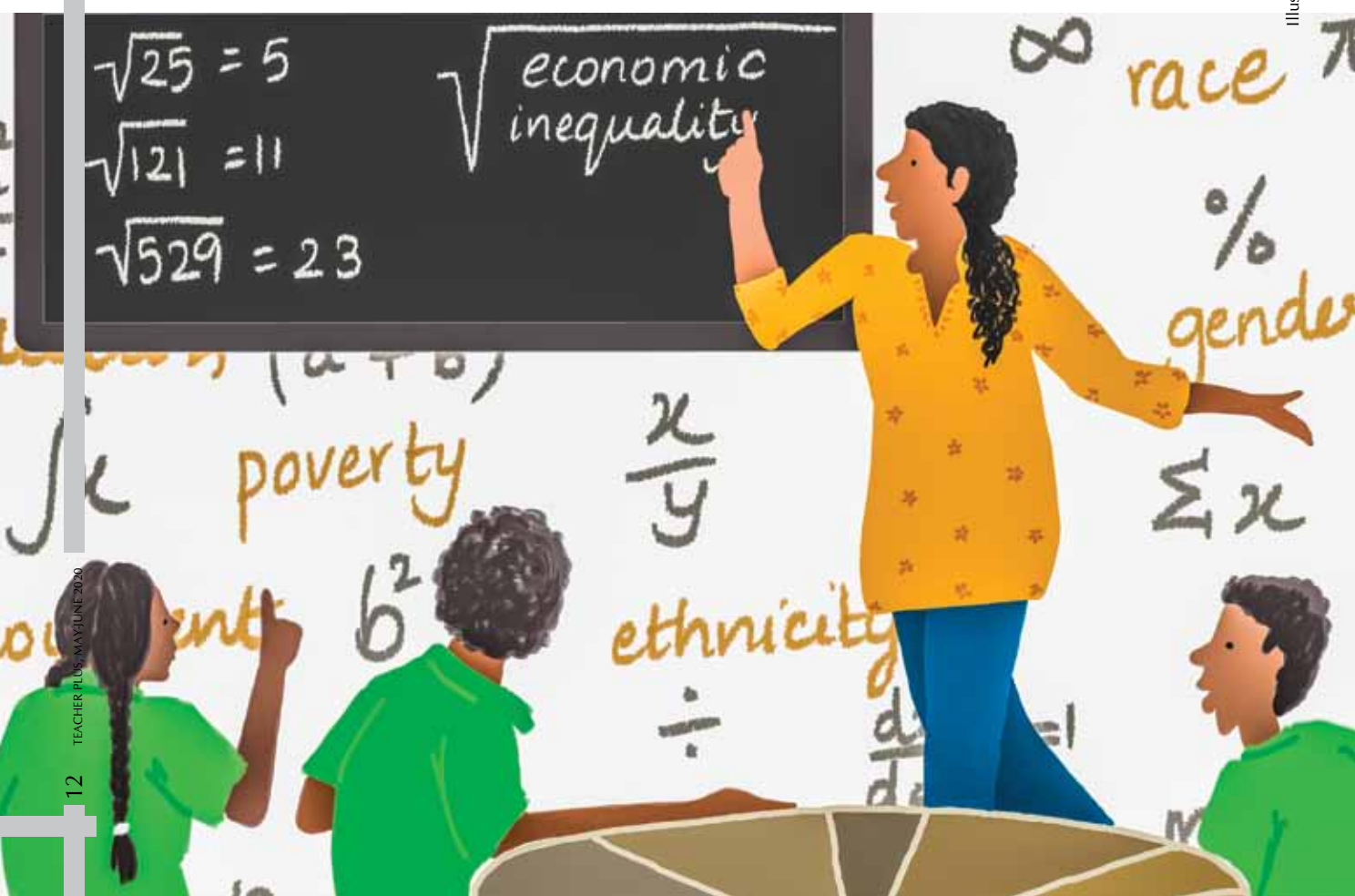
and to some of my peers. It helps me appreciate the connection between mathematics and problem-solving skills at large.

How can a mathematics curriculum incorporate issues related to social justice? There are multiple examples and one of those has to do with "studying probability in the context of a unit on how the lottery increases the economic divide between rich and poor." According to Osler, such a unit can enable students to learn not only the mathematical skills they need but also about economic inequality in the society they live in. They can use their mathematical knowledge to brainstorm ideas about solving real-world problems.

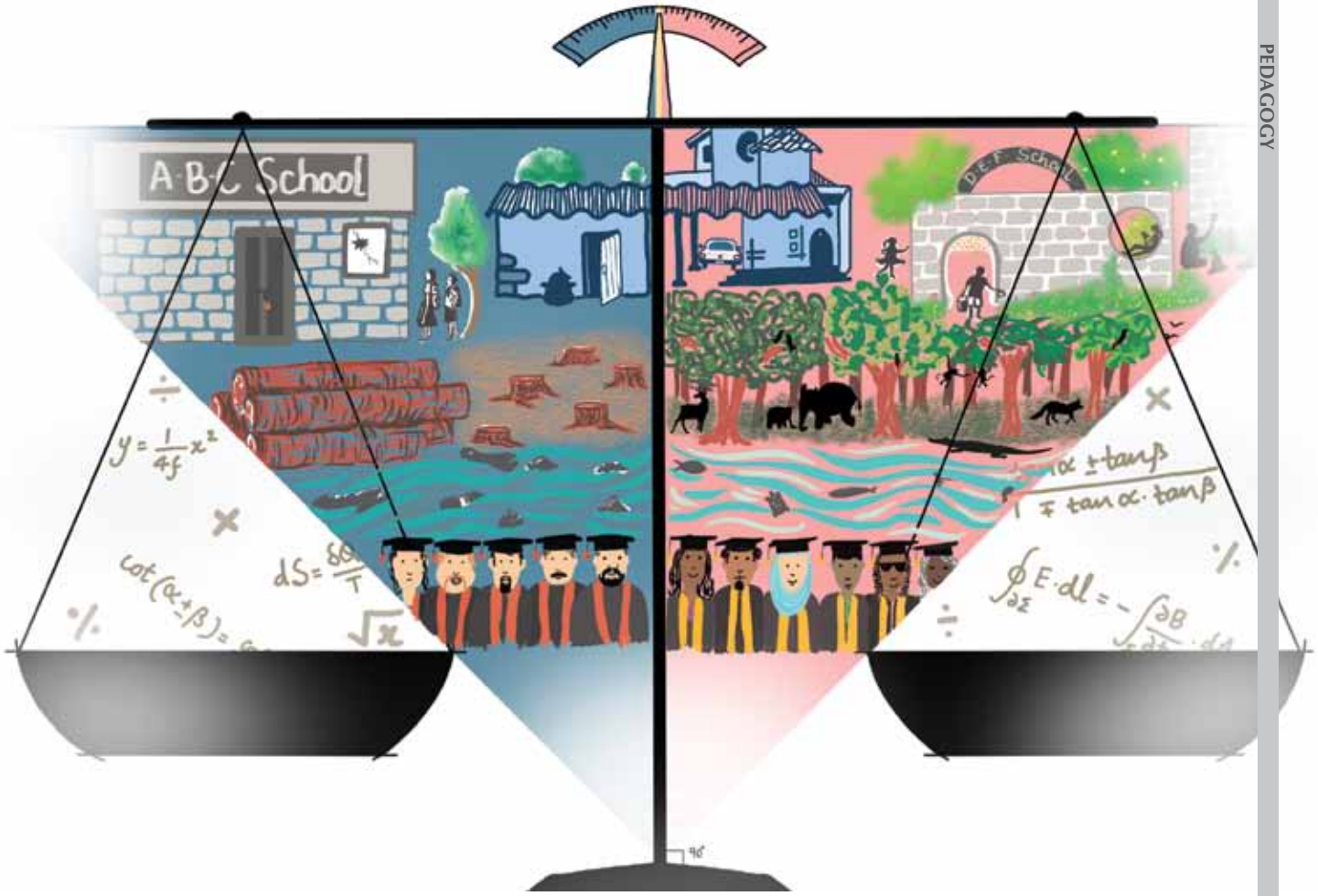
What opportunities can be carved out in your school for this kind of learning? Look carefully at the news reports you are reading and make a note of the numerical data that can be used in your mathematics class to hone skills such as critical thinking, logical deduction and quantitative reasoning – all of which are mentioned in the Draft National Education Policy of 2019. Osler's examples include defence budgets, worker salaries, population growth, racial profiling, waste disposal, predatory lending, etc. How do you feel about designing mathematics classes using numerical data about casualties, disease prevention, food distribution, and economic losses, being gathered and circulated in the context of the COVID-19 pandemic?

If you are wondering why mathematics teachers should concern themselves with social justice, this is what Osler has to say: "Math was behind the development of nuclear weapons. It is used to maintain an economic divide between a handful of wealthy, White people and the billions of poor people of colour around the world. It is used as a rationale for depriving people of access to cheap, life-saving drugs. So my question is: what good has the progress of mathematics as an intellectual discipline done for people?" Osler believes that if mathematicians had been trained to think about social justice, there would be less suffering in the world.

What is important to note here is the author's absolute insistence that students should not feel short-changed as far as the mathematics content is concerned. Denying young people the opportunity to learn mathematics because the teacher is more interested in discussing certain social issues



Illustrations: Swetha G Nambiar



is unacceptable to him. This passionate stance comes from his awareness that mathematical literacy is often a gateway to opportunities for higher education. Without it, they are unlikely to perform well on standardized tests for admissions.

There are other questions that you might want to think about as an educator. How would students' parents respond to this approach to learning? What could be done to ensure that students do not end up feeling helpless in the face of systemic problems but adopt a solution-oriented lens? How would assessment capture the skills and content knowledge you are looking to test? What could be some avenues for students to put their learning into practice and see tangible change in their communities? How can you prepare them to think critically without indoctrinating them with your own belief system?

Osler also recommends a few educators and scholars that you could look up if this mandate of integrating issues of social and economic justice into the mathematics curriculum has caught your interest. These include Rico Gutstein, Bob Moses, William F. Tate, Danny Bernard Martin, Arthur Powell, Marilyn Frankenstein, Bob Peterson, Gloria Ladson-Billings, Paulo Freire, and Henry Giroux, amongst others. I would add Sara N. Hottinger, Margot Lee Shetterly, Anita Wager, David Stinson, Rochelle Gutiérrez, David Richard Wagner and Parvin Sinclair to this list. That is a lot to get you started. Enjoy!

TP

Pallankuzhi – a school math project

Jayasree S and R. Ramanujam

Jayasree S is a teacher, currently registered for a PhD at HomiBhabha Centre for Science Education, (HBCSE) Mumbai. She can be reached at <jsree.ts@gmail.com>.

R Ramanujam is a theotetical computer scientist at The Institute of Mathematical Sciences, Chennai. He is also interested in science and mathematics education and popularization. He can be reached at <jam@imsc.res.in>.

Consider two children absorbed in a board game. They are focussed, thinking, strategizing and trying to execute plans. All this is worthwhile and children are encouraged to play board games. Does such play also help their *mathematization abilities*? By itself, perhaps not, with some guidance and help from a teacher, perhaps yes. In this article we would like to suggest that such an exploration is worthwhile for teacher and student alike; it is certainly joyful.

We recently had the opportunity to work with a group of 9th graders in a matriculation school in Chennai on their annual science project. This group of students wanted to dig into 'ancient mathematics' and specifically into 'ancient mathematical games'. While we read about many such games and played them, here we share some of our learning via **pallankuzhi**, the pits and pebbles game.

The game description

The version of pallankuzhi we played is a two-player game, with a board containing seven pits on each side, and six pebbles in each pit initially.

Image Courtesy: https://upload.wikimedia.org/wikipedia/commons/6/64/Pallankuzhi.JPG

Each player owns the seven pits on her side The first player starts the game by choosing one of the pits on her side and redistributing (sowing) the pebbles there in, one pebble per pit, starting from the adjacent pit and traversing the board in an anti-clockwise direction. Once all the pebbles are 'sown', she draws on the pebbles from the pit adjacent to where she dropped the last pebble and continues the 'sowing' till such time that the pit from which the pebbles are to be drawn is empty. In this case, the player captures (harvests) all the pebbles from the next pit – i.e., if the pit next to the *last drop* is empty; the player captures all the pebbles in the next pit (empty or not) and the turn ends. The turn now passes to the next player, who chooses a pit on her side and continues as above. The game continues till all the pits owned by one player are empty on her turn. The goal of the game is to maximize the number of pebbles captured. In general, the game is played with cowrie shells or pebbles or locally available seeds.

Pallankuzhi – a part of culture and tradition

Pallankuzhi being part of the cultural heritage of Tamil Nadu, most of us had played it at some point in our lives and had a game board available, though no longer in regular use. The students, in conversation with their parents and grandparents, realized that this game was part of the elders' growing-up as well and that they knew a trick or two that would yield a rich 'harvest' of pebbles. Some students talked of people who,

taking into consideration the current distribution of the board, could use some quick counting strategies to come up with moves leading to a capture. This 'look-ahead' in the game could involve being able to visualize multiple rounds of sowing and therefore a certain mathematical acumen.

In the South Indian states, the game is largely played by children and womenfolk. There are traditions and taboos surrounding the game, like the game being played on certain 'auspicious days'; that the game should be played in the daylight hours only, etc. There are mentions of the game being a recreational activity for agricultural labourers during hot afternoons and of the game being played with currency coins instead of pebbles and seeds soon after a wedding in some communities (Balambal, 2005). The presence of rock engraved game boards in temples points to the role of the game in community life. Balambal (2005) and Murray (1952) list different versions of the game as played in different parts of India and abroad. These include solitaire versions, triangular boards and boards with five, six or seven pits.

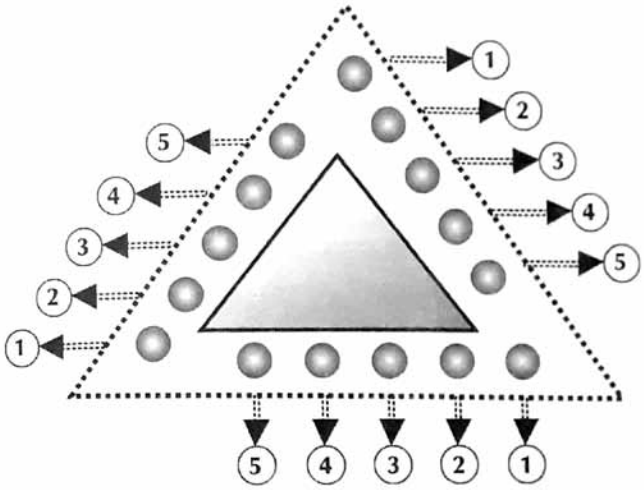


Image from Balambal (2005) p 99

Pallankuzhi variants from across the world

The family of 'pit and pebble games', usually termed *Mancala*, all involving sowing and harvesting of identical pebbles, with many distinct rules of play, is quite widely spread geographically and temporally. The types of game boards also vary – rock-carved boards to pits dug in sand and ones made from material as varied as wood to ivory inlaid with gold. Archaeological findings from sites such as the Karnak Temple in Egypt, Pyramids of Meroe, Palymra in Syria, Theseum in Athens indicate the presence of the game from the beginnings of human civilization. You may want to look at some references listed at the end of this article for details.

The game has many connotations. In many communities, the game has been associated with rituals and festivities around invoking rain, celebrating a harvest, birth and death, etc. In parts of Africa it is a "noisy social occasion with cheers and jeers, laughter and occasionally anger and brawling". Though a two-player game it often takes on the face of a team game, with the boundaries between players and spectators blurring. Among some African tribes the chieftain is chosen by playing a game of Mancala. In most African countries Mancala is a game *only* for men, unlike in India. Townshend (1979) hypothesizes that a woman defeating a man at a public game, or ridiculing the loser, would be a dent to male power. The game is considered a means to educate young men on the social values embodied by the elders and not learning to play would be considered anti-social.

A peek into the classroom – whiffs of mathematization

We now turn to the classroom and hope to highlight the opportunities for mathematization afforded by the game. A group of five students from class 9 took up this exploration.

Even as students started to play the game, there were differences in *rules of play* that each student came up with in consultation with knowledgeable elders at home. One of them said that the initial configuration has to be six seeds per pit and another said it has to be five, and that one could harvest the seeds whenever the count of seeds in a pit (on one's own side) reaches three. We decided to go with the six seeds per pit version, hoping to see if we could observe some patterns that could lead to a strategy to win. Soon we were struggling to keep track of the number of seeds in each pit and counting on to see what configuration of the board would eventually result after a turn. We then decided to play a three seeds per pit version of the game, following Polya's heuristic of ***'simplifying the problem'***. We also had to ***clearly define the problem***, by spelling out the rules of the game – like what constitutes a turn, and when does it end, what are the rules for capture, which pit do we draw from once the seeds in hand have been redistributed and so on.

When we found even the three seeds per pit version hard to keep track of, we decided to simplify even further and strip down to a

seemingly trivial configuration of one seed per pit.

With this some patterns started becoming obvious and students came up with statements like "In this game, we could capture two seeds from the pit adjacent to the one we started from." Soon the students wanted to see if there is a pattern to be observed in the two seeds per pit and three seeds per pit versions, more specifically if they could make predictions about the opening move. With two seeds per pit, the position of the pit from which one could harvest given a starting pit was obvious to them – the pits were moving in tandem – one step at a time. But *verbalizing* the relationship proved to be a challenge.

The suggestion to ***number the pits*** gave them a better way of talking about these relationships. Now they could make statements such as: "In a one seed per pit game, if the first player starts with an odd-numbered pit, the second player has the option of starting from the even numbered pits on her side." This also gave them a way of listing out the sequence of pits from which seeds would be drawn to be redistributed. Playing with these sequences of numbers helped them predict the course of the game *without actually playing it*. This constitutes a form of mathematization in itself.

For example, the sequence 1, 4, 7, 10, 13, 2, 6, 10 – is the sequence of pits from which seeds would be drawn for redistribution starting with pit 1 when there are two seeds per pit, (pits are numbered 1-14 clockwise). The two seeds in pit 1 get redistributed to pits 2 and 3 and the seeds in pit 4 are drawn and dropped in pits 5 and 6 and then the seeds are drawn from pit 7 and so on. Coming to pit 13, the seeds get dropped in pits 14 and 1, but now there are three seeds to be picked up in pit 2, and so the next draw would be from pit 6. Continuing thus, the game reaches a stage where one needs to draw from pit 10. But since pit 10 was drawn from in the previous round, it is empty and so the player gets to capture the seeds in pit 11. ***Writing out these sequences of numbers gave them a way of verbalizing the patterns they observed and made the patterns more obvious as well.***

Moving beyond the first move, the second player has a number of options to choose from – she could play any of the non-empty pits on her side. So how does one choose between the options? We knew that we should maximize the number of seeds captured. So then we calculated the seeds that could be captured with each available option and looked for ways to represent this information. A tree-diagram came in handy.

Options in the game represented as a tree diagram on the floor. (Image courtesy: Vinusha Venkatesh)

A notation was evolved to represent the configuration of the board at the end of each turn as well. While we are nowhere near finding a winning strategy (obviously!) the exercise has opened up a number of possibilities and points of departure for initiating many more school projects across disciplines.

Representations of Board Positions (Image courtesy: Vinusha Venkatesh)

Points of departure

Some students might want to go on a historical journey, digging into the origins of the game. Marking the areas of prevalence of the game on a map could be an exercise in geography. Comparing the games as played in two different regions and looking for similarities and differences might give an indication of aspects of the game that could be varied. Letting students come up with their own variations of the game could be an interesting and useful exercise. This could lead to multiple questions – For example, the six-seeds-per-pit game that we started out with and the one-seed-per-pit game that we eventually ended up exploring – are they different games or is one a simplified version of another? Varying which dimensions of the game would constitute a 'simplification' and which dimensions would result in an altogether different game?

Patterns observed for a three-seed-per-pit game or a two-pits-per-side game may be appropriately extended to a six-seeds-per-pit or eight-pits-per-side games. Having fewer seeds per pit or fewer pits per side are ways of simplifying the game.

However in our game, if we had chosen to

- redistribute the seeds from the pit of the last drop instead of the adjacent pit, or
- if we had insisted that a 'turn' would end when the seeds drawn from the first pit chosen are redistributed

these games would require a different analysis. So this leads us to the question – change in what aspects of the game call for a fresh analysis?

Though questions related to a winning strategy for a given set of rules and configuration of the board are difficult to answer, students can still investigate simpler questions like:

- how many different configurations are possible, after say three moves, given an initial configuration, or
- does a particular configuration give an advantage to one of the players, or
- are there impossible configurations, and so on.

(The configurations have to be chosen carefully for the questions to be answerable).

The game apart, even investigating the number of ways a given number of seeds can be placed in a given board can be a good exercise in combinatorics (see References at the end).

Reflection

We suggested at the outset that board games of strategy may provide an arena for guided mathematization. Apart from the attractiveness of pallankuzhi being culturally rooted, we note that it has a low threshold (unlike, say, chess) and everyone playing the game can formulate some strategies. Can it lead to complex mathematics? We do not know but this seems worth exploring. (For instance, is there a Sprague-Grundy type theorem for pallankuzhi?) The fact that pallankuzhi offers multiple starting points, branching pathways for exploration and scope for a local formal language for articulation of strategies strengthens our belief in its scope for mathematical exploration. Understanding how teachers' guidance can shape such exploration requires further work and sharing of experience among teachers.

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Hands on paper, hands-on geometry!

Roopika Sood

The author has taught English, social studies and origami for nine years in Rishi Valley School (Chittoor) and Sahyadri School (Pune). Thereafter, she headed an educational non-profit organization that delivered digital educational resources for government school learning and teacher training in Chennai. She also teaches Theatre and Advanced English to children and consults with several schools in Chennai over school development projects and language teacher training. She can be reached at <roopika.s@gmail.com>.

(This article is a reflection on the author's experiments with origami in a boarding school as part of a Makerspace and how curriculum and pedagogy in origami-based geometry organically developed after children stumbled upon symmetry, form and spatial understanding while folding paper models. While origami began as an art in this school, it soon enthused children to explore the math behind origami through their own tinkering experiences. This article brings to light experiences where experiential learning preceded technical learning in geometry across all ages.)

It was in June 2014 that the old staffroom at Sahyadri School (KFI), Pune had been transformed into a space that six adults had envisaged as the new space for children and adults. Organically, it was called 'Tinker Shed' – a space for tinkering with materials. Since it was still developing in form, structure and method, it was more a shed with tools and materials, a warehouse, rather than a workshop or a carefully organized laboratory. We wanted it to be the beating heart of this beautiful school – we conjured up images of a lively lake with various busy creatures. Little did we anticipate the tsunami of personal and group experiences which were a turning point for many of us in how we understood hands-on learning!

One of the many activities in its ambit was origami – an unusual choice for a tinkering space. However, we were convinced that indulging in origami could only be a good thing for children. This was particularly so for our children were living in a boarding school, indoor-bound for weeks thanks to the monsoons that flood out any outdoor activities even on a hill. In its first year itself, we decided to have everyone play with paper – from pre-school to grade 10 (the highest class in the school that year)!

Photos courtesy: Roopika Sood

We organized multiple photocopies of instructions from origami books and online resources, theme-wise and with varying difficulty levels. We organized paper of different colours in different sizes, scissors, paper cutters, cutting mats and glue at arm's length. Very soon, we realized that this was becoming a hotbed of folding fads for the 300 students on

campus. With a superb value-for-effort quotient, origami classes ended with walking out with mini trophies to showcase an hour spent with focus, determination, visualization and dexterity skills. "I want to make that" became the buzzword and soon we had a growing art gallery of physical models which was like a menu card for children to choose from!

Origami and symmetry

"It just looks odd to me," said a rather confused eight-year old girl, holding her six-unit cube, unsure of what was wrong with it, even though it very much resembled a dice. Instead of using three colours for each of the three dimensions, she had used two colours, and therefore some of the six faces, in her words, seemed "odd". Over the next 30 minutes or more, she sat back and took her own time to correct her model – using different colours, but same size of sheets of paper – working with trial and error, to begin with, and then finally understanding what colour-symmetry this particular model demanded. She walked out of the Origami Room, exuberantly and loudly admiring her own piece of art. To anyone else, it was, just a cube.

To me, as her origami teacher, it was reaffirmation of origami being a tool to fan what is innate in all my children – and remains latent in many an adult – the curiosity to solve a puzzle.

For the uninitiated, origami is the art of paper folding – no cuts, no glue. It is traditionally associated with the Japanese where it originated, but modern origamists employ several advanced techniques of paper-craft such as sculpting, shaping, curved folding, wet-folding, etc. Symmetry, in its simplest definition, is 'the quality of being made up of exactly similar parts facing each other or around an axis'. Symmetry is fundamental to origami.

Several recurring incidents of origami students watching their models carefully and checking for colour and form symmetry convinced me to keep the following objectives at the fore in formulating a learning ladder combining the study of symmetry through paper-folding constructions. At the end of the module, the students should have been able to:

1. Calculate the colour symmetry befitting the model at hand.
2. Modify the size of a model mathematically – working with proportions, dimensions and basic geometry.
3. See geometric symmetry in models that look more art than math, in particular, the case of *kusudamas* (decorative floral balls).

4. Understand basic kinds of symmetry without knowing the terminology.
5. Stay keen on pursuing the puzzle-like joy of working out asymmetries.

What is now a robust, tested module on teaching symmetry rules through origami constructions, took its own time to be fine-tuned. What was very interesting was to see the left-brainers and the right-brainers point out stereotypical asymmetries while working. Such conversations were dotted with geometry vocabulary that had become common parlance.

Origami and 2D geometry

"Fold it into a triangle first and cut off the rectangle."

For a nine-year old to give that instruction while teaching her classmate to cut a square from an A4 sheet may seem tutored. However, the tinker shed was a hub for words that would make a passer-by think that we were revising a lesson on quadrilaterals. A week-long experiment raised the bar – each student had to teach a model to another but not by demonstration, only verbally.

This meant that while teaching something as simple as the Yoshizawa butterfly, one had to use words like triangle, diagonal, rectangle, right angle, intersecting lines, perpendicular, etc. The next challenge would be to verbally demonstrate the most famous origami symbol – the Japanese crane. How would you do without saying rhombus, mid-point, perpendicular and isosceles triangle?

Origami and 3D geometry

"This one is also another of the star-shaped ball kind, akka?"

This was a common question when my young origamists wanted to fold various three-dimensional models based on dodecahedrons. Kusudamas are decorative floral balls, often made with the dodecahedron or the icosahedron as the base shape. The cube was the most common and often lost its charm in the jungle adventures through middle-school ambitions. Over time, we could explore all of the Platonic solids and many of the Archimedean solids through origami ambitions. It was the most organic way of understanding the number of edges, vertices and shapes of faces for each of these solids.

There were children as young as grade 5 talking about 'stellated' and 'truncated' solids. There were students in grade 7 and 11 wanting to working on fractals. There were students from grade 9 onwards who were fascinated with planars when they saw "paper-slices intersecting at the centre". As a teacher, I could not have asked for a more fast-tracked and steeper learning curve into geometry!

Origami and visual puzzles

"Fold the square into two rectangles and similarly, fold the sheet into 16 equal parts."

The result of this seemingly straightforward instruction was astounding – the paradigms that this showcased got me hooked on to origami folding puzzles. My class of eight students had folded their original square sheets into equal parts but only three of them had folded 16 squares. I wouldn't want to reveal the other possibilities here for the readers to tinker on their own! Here's a hint, though – the class revealed that there were "infinite ways of dividing a square into 16 equal parts!"

- How would you fold the largest octagon from a square sheet of paper?
- How many isosceles triangles does this octagon have?
- Could you make a rectangle from a piece of paper that does not have any straight edge?
- How would you make an equilateral triangle from a circular paper?
- How would you fold a square that has exactly half the area of the square sheet of paper?

Such were the questions that my students would love coming up with while tinkering with paper – there is nothing more exciting than knowing that setting a mind-boggler for a large population of boarding school is democratic – anyone could come up with these puzzles and I promised to take these to every class that I conducted. For a while, we would put these up as charts outside the Tinker Shed and they became heavily competed over – not just for solving the mystery but also for setting the next puzzle!

The road ahead – the origami universe at large

Origami is not mathematics, origami is not visual arts, origami is not craft – it has elements of each and a lot more. The study of polyhedra became as much a study in art – and that is what I consider the greatest success of the origami program. There have been several origamists and mathematicians who have stumbled over the umbilical cord that connects the two and much research has gone into exploring other areas of mathematics that lend themselves to origami and vice-versa. The story here is only a nudge to readers to blur the lines between math and origami and enjoy the process of hands on paper, hands-on learning and child-like tinkering!

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Illustration: Sunil Chawdiker

Abstract to concrete – biology to the aid of mathematics

Geetha Iyer

The author is a consultant for science and environment education. She can be reached at <scopsowl@gmail.com>.

"… if you are faced by a difficulty or a controversy in science, an ounce of algebra is worth a ton of verbal argument," – ***J B S Haldane***, biological scientist

Corona virus! A piece of RNA wrapped in a coat of proteins and fat is wreaking havoc on earth in a manner quite unfamiliar to lay humans! To understand its life-cycle, spread, infective rate

and effect on human systems, epidemiologists, researchers, doctors, in short everyone in the scientific community is turning to maths! Can there be a better evidence of the importance of maths to life science – *aka* biology?

But high school or even middle school mathematics rarely use examples from the world of biology or highlight the close relationship between the two. I argue that it is the use of examples far removed from a child's world that is responsible for the fear of maths. Experience is necessary to build knowledge. Knowledge reduces unknown fear. As COVID-19 spread its fangs (more precisely, its protein coat) it is math that very graphically (pun intended) has helped provide evidence of the gravity of the situation, so people could accept curfews and shut downs. Examples if sourced from the world of biology can provide experiences to a child that will help build mathematical knowledge.

The deep entwining of the two disciplines will become vivid if we examine our own life. Take a moment to dwell on the life of a human being from birth till the very last breath. It will make no sense without mathematics.

Maths in human lifetime

What is the weight of the new born baby? Has she doubled her birth weight at the end of three months? How many times do you nurse her? How many spoons of cereal powder to be added and to how much (i.e., what volume) water? How many teeth has she sprouted? Three years and the question moves to – how far is the school from your home? And then at school – How many fingers on each hand? How tall are you? How many slices of bread/idlis did you have for breakfast?

Get the picture? Life moves on the wheels of mathematics. Next when the young child – with an oversized bag on her back- enters school, the first lessons in maths use materials from the biological world. Seeds to make sense of numbers. I remember my lessons on odd and even. There were three different games using tamarind seeds or *manjadi* (*Adenanthera pavonina*[1]) seeds to master the concept. Games that I still love to play at 65. As we move into the teens it's about how tall or how short, what size or weight, or all about calories in food!

When I catch the flu (or the dreaded COVID-19) the doctor reads the temperature. 102°C. Hmm. What medicine should I take? 2.5mg of Crocin. This tablet pack says 5mg. So take half a tablet. How easily the fraction is done isn't it? Into middle age and attention shifts to questions of the heart, liver, etc. The blood pressure is 120/80 mm Hg but why is my heart beat more than 72 beats/minute? Lying in a hospital watching the doodles on the monitor, our mind is busy counting or calculating a number of things. Maths is with us till the very last breath. In agriculture, nutrition, cooking and medicine, math is the tool we use hardly being aware of it.

Biology's influence on maths

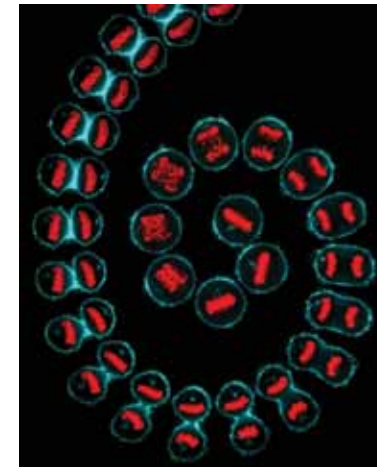
From Aristotle to Francis Crick, biology has contributed immensely to the advancement of maths.

French mathematician Benoit Mandelbrot described fractal geometry by observing patterns in nature such as branches of trees, growth of plants, etc.

Robert Brown, the famous botanist discovered the Brownian motion while observing pollen grains suspended in water. Mathematical description for the motion developed by the American mathematician Weiner is an important part of the probability theory. The Weiner equation is used to study a variety of scientific processes. Mathematics has advanced in leaps and bounds through the study of nonlinear equations of diffusion influenced by processes from biology.

Cormack and Hounsfield received the Nobel Prize in physiology and medicine in 1979 for their theoretical mathematics that made possible image reconstruction by CT scan. MRI and PET are similar instances of humanity benefitting when mathematics and biological sciences twined.

Fibonacci sequence or Fibonacci number is described as nature's universal code. So common in nature, it may be seen in something as small as cells in organs image of cancer cells dividing, in the cells of the ovary of an Angler fish[2] to something as large as inheritance patterns in humans. The number of petals in



a flower, arrangement of rows of seeds in a sunflower, scales in a pine cone and the most famous of all – spirals in sea shells, nature has adopted this sequence of number so frequently leading one to wonder the reason behind it. Mathematicians believe that it facilitates efficient flow of information.

Although modern history attributes the discovery of this sequence to the Italian mathematician Leonardo Fibonacci, it was an ancient Indian Jain scholar, poet and polymath – Acharya Hemachandra[3] – who described the Fibonacci sequence 50 years before the Italian. He discovered it while studying the rhythms of ancient Sanskrit poetry.

Geometry in life

Biological structures lend themselves to the study of geometry. A virus is nothing but a protein coat that encloses its genetic material. This coat often has complex geometrical shapes. For example, where the protein coat has a shape with 20 sides it is in the form of an icosahedral; some are helical. In the case of the corona virus it is spherical. Some plant cells have a very ordered hexagonal shape. Protozoan called Radiolaria that live in the sea are unicellular organisms but the geometry of their skeleton is amazing. Math teachers could use these examples while teaching geometry.

The field of topology is the mathematical study of shapes. Structures such as the brain fascinates mathematicians. Brain mapping uses hyperbolic and spherical geometry.

Fractal geometry. One of the most fascinating aspects of geometry of nature, observable in the branching patterns of trees, vascular tissues in our body, in shapes of all kinds in nature. Unfortunately neither biology nor math teachers talk about this in their classes.

Scutoid[4] – Scientists and Engineers from three universities used principles of geometry to create a new shape called scutoid. They found out that this new shape was already in existence in nature long before they named it. When cells have to form an organ, they reshape and pack themselves closely, and the shape they take on is this geometric solid called scutoid.

Classroom examples

Primary math curriculum incorporates examples from the natural world. The same cannot be said for middle or high school math. As a student moves into middle school, math concepts increasingly require abstract visualization. Here is where difficulties begin for many. Examples from the biological world can assist in overcoming difficulties especially in topics from algebra and geometry.

Events from biological processes can be used to introduce a topic. Then data from these processes can be used for teaching or setting problems and equations to solve. Given below are a few samples – outlines of topics which should catch your pupil's attention. (*Refer to the Teacher Plus website, www.teacherplus.org for more such exercises.*)

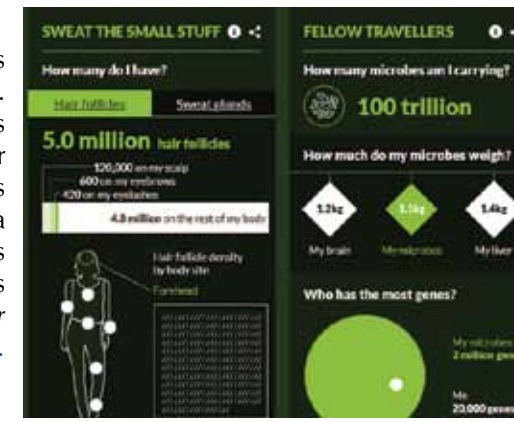
The information given below may be used while teaching averages, percentages, decimal conversion, algebraic equations, weight-volume relationships, fractions, ratio and proportion, orders of magnitude and so on.

1. A fun example would be this from the digestive system – Flatulence (or passing wind/ farting) and burping are two processes that tickle children a lot. Maths problems related to these processes would not only serve to make math fun and easy to learn but also create awareness about food eaten.

Info: Many chewing gums and some diet sodas contain sugar alcohol that can cause gas to build up in the intestine. While chewing gum, people tend to swallow air that can get trapped in the gut, leading to burping. Onions contain fructose which is broken down in the intestine. This breakdown causes gas to form with an odour.

Data: On an average human beings produce 0.47-1.41litre of gas/day and pass it out 18 times/ day. The data may be used while teaching algebra.

2. Here is some data calculated for an adult female aged 60+. Enjoy using them.



You can also feed in some basic information about yourself and find out what you are made of. Try it, it's fun[5].

A project for the class

Here is something that will make learning linear equations memorable. The example given below may be used separately or can be part of a project on linear equations.

Everyone studies biology till class 10 and therefore knows that William Harvey[6] was responsible for the discovery of the circulation of blood. How he did is an interesting story, too long to be told here. He had no instruments or even anaesthesia to open the body and look at the functioning of the heart. He turned to mathematics. Through some careful measurements and calculations he was able to disprove the existing theory and explain how blood circulated in the body. The discovery and data he used can be a fascinating example for teaching linear equations.

Linear equation is the relationship between two or more variables. The relationship between the volume of blood pumped from the heart during each heart beat(V), the no. of heart beats per unit of time(R) and the amount of blood pumped by the heart through the body per unit of time (C) can be represented in a liner equation, C = V.R. This is what Harvey found out, calculated and estimated that the average human had only 5.5 litres of blood. (For more details on Harvey's work refer to the TP website)

Teachers can give a value for cardiac output and then ask students to calculate the length of time taken for the entire volume of blood to circulate once through the body. This is the simplest one. Many more exercises are possible using data from the human cardiovascular system.

Examples from photosynthesis, respiration, heights of students compared with the lengths of their large bones may also be used for the project.

Ancient cultures did not fragment knowledge into sciences or mathematics. There was just one discipline called philosophy. These divisions came about as knowledge of the natural world increased. But, the time has come to dissolve some of these divisions.

In this emergent new society that seeks to inhabit the moon and mars, which is seeking to find extra-terrestrial life and produce human life in a petri-dish, biological sciences have entered an even greater exciting phase than when the human genome project began. Math is now a kind of microscope for biologists. It helps them discover the invisible in the world of living beings. When traditional laboratory experiments fail to find answers, biologists turn to math to model the systems that they are studying, such as loss of habitats, extinction events, brain functions, etc. Mathematical biology[7] has emerged as a separate discipline in universities. Math curriculum at the high school too, must incorporate these developments. Education of students would be incomplete if they learn maths in isolation.

If mathematics and biology teachers are still not convinced of these connections, I recommend they read this most enlightening book by a mathematical biologist D'Arcy Wentworth Thompson titled *On Growth and Form*[8].

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The joys of solving problems

Shailesh Shirali

The author is the Director of Sahyadri School KFI. He has been in the field of mathematics education for many decades. He is the author of several mathematics books for students and teachers and serves as Chief Editor of the magazine At Right Angles. He has a deep interest in nurturing inquiry into fundamental issues among teachers and students at the school level. He can be reached at <shailesh.shirali@gmail.com>.

Mathematics as a discipline has many characteristics, but none may be as central to the subject as problem solving; none may be as evocative and memorable; and, perhaps, no other activity holds so much scope for the thrill of discovery. If we as teachers are to convey and instill a lasting love of mathematics in our students, then we must exploit to the fullest extent the possibilities offered by this activity.

One may view problem solving as a playful game of tease – a game between the teacher and the student, or between the math enthusiast and the subject itself. The element of play is essential. Here we use the word 'play' in its original sense and not the sense it has acquired in modern times: allowing free movement and exploration, with no compulsion, no sense of "I must prove myself" or "I must show that I am better than you."

The reader may wonder whether we are making a big deal out of this; after all, most people would imagine that school mathematics is *all* about solving problems. That certainly is the impression that children and parents would carry with them from their experience at school. However, these 'problems' are merely back-of-the-chapter exercises; they cannot properly be called problems at all – they are merely drill exercises. To be taught, say, the quadratic formula for finding the solutions of a quadratic equation, and then to apply the formula to 10 different quadratic equations, differing only in small ways from each other: that is not problem solving at all! That is merely doing a set of exercises to reinforce the algorithm, and 'drill' is a more appropriate word for such an activity, not 'problem solving.' No, what we are referring to in this article is not this but the *culture of solving non-routine problems*.

No article about problem solving can leave out the incisive observations made by the mathematician George Pólya, who has written with such eloquence and clarity on this topic. Here is one quote of his that is absolutely on the dot:

It is better to solve one problem [in] five different ways, than to solve five problems [in] one way.

We will have occasion to share more quotes from Pólya later in the article.

The title of this article refers to the 'joy' of problem solving, but that does not mean that one does not feel pain during the process! Quite often, the dominant feeling during problem solving is one of frustration, of feeling rather stupid. Solving a problem in geometry, you stare at the figure, and then you stare more, and still more, all the while feeling that you have run into a wall. And then, all of a sudden, a glimmer of an idea strikes you, maybe while out on a walk, maybe while you are in bed, trying to fall asleep, or when you wake up in the morning, and the idea stays with you. You continue chewing on it, off and on, and, bit

by bit, the pieces start to fall in place, like a jigsaw puzzle. Then, one day, the picture is complete, and you gaze at it in amazement, marvelling at its beauty, and wondering why you never got the central idea in the first place. This, more or less, is the experience of problem solving. Would not it be wonderful if every student could have a taste of this in school? Is it not part of our work that every student experiences this taste?

A peculiar syndrome that confounds Indian education is that of "completing the syllabus." We must, absolutely must complete the syllabus – and woe to the teacher who does not! This phenomenon brings out a dichotomy we see sharply highlighted in education systems all around the world and particularly so in India: focusing on *what* to think, rather than *how* to think. We tend to be quite obsessed with the 'what' and on ensuring that every little detail in the syllabus is attended to, down to the last comma and full stop, but curiously neglect the 'how', which surely is the most essential part of education, for the 'how' is about learning *how to think*. What better way is there to learn the 'how' than to actively plunge into problem solving? We quote Pólya again:

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet … and, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe and to imitate what other people do when solving problems, and, finally, you learn to do problems by doing them.

One can learn how to think only by actually doing so and finding what it is all about; there is no technique for it….

However, it is important that the teacher not be carried away in his enthusiasm when it comes to posing challenges for the student. He must assess for himself as accurately as possible the level that is most appropriate to the student. If he poses challenges that are too easy, that are not challenges at all, he runs the risk of creating a dismissive attitude in the student, a lack of engagement, a lack of creative tension. If he poses challenges that are too challenging and much beyond the ability of the student, he runs the risk of creating a mental block in the child, a lasting feeling that "this subject is not for me," or a permanent impression of inferiority. As the mathematician David Hilbert noted,

A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts.

There is a fine line to tread here. We need to find that line by careful experimentation and by getting to know the child as well as possible – and that can happen only through close contact between teacher and child.

At the primary and upper primary level, problems involving numbers are available in plenty. Children love such problems! For example, there is the whole family of problems called *cryptarithms*, which are simply coded forms of arithmetic problems with obvious restrictions such as: (a) two different letters cannot represent the same digit; (b) numbers are not permitted to start with the digit 0.

A particularly simple instance of such a problem is the following. Find the digits O, N, G if

$$ON + ON + ON + ON = GO.$$

It turns out that this has a unique solution, but one must not say this in advance; *children need to discover it on their own*. However, a tiny change gives rise to interesting effects. The following problem has multiple solutions (and children should find them on their own):

$$ON + ON + ON = GO.$$

In contrast, the following has no solutions at all:

$$ON + ON + ON + ON + ON = GO.$$

These are interesting discoveries for children to make for themselves – that some problems have multiple solutions, while others have no solution. All these phenomena have parallels in real mathematical applications. Here is a well-known example: it is not possible to find positive integers a and b such that $a^2 = 2b^2$ (this is just another way of saying that the square root of 2 is an irrational number). A more spectacular and famous example is Fermat's Last Theorem ('FLT'), which is the statement that there do not exist positive integers a, b, c and $n > 2$ such that $a^n + b^n = c^n$.

Here are two cryptarithms that I have tried out with many batches of students.

1. Find the digits A, B, C, D given that $ABCD \times 4 = DCBA$. (This means that when we multiply the four-digit number $ABCD$ by 4, its digits are reversed.) A related cryptarithm is $ABCD \times 9 = DCBA$.

2. Find the digits T, W, O, H, R, E given that $TWO \times TWO = THREE$.

I have experienced greater class engagement from the first one (the digit reversal problem) than from any other non-routine problem! Perhaps this is because the code can be unraveled using pure mathematical reasoning. There is something extremely appealing about being asked to find a four-digit number whose digits are reversed under multiplication by four. The utter simplicity of the statement is part of its appeal.

Another rich source of non-routine problems is *magic squares*. We can start with questions such as, "Why do we not have 2 x 2 magic squares?" Then we move to 3 x 3 magic squares. It turns out that using the numbers from 1 to 9, just one magic square can be made; every other magic square using these numbers is simply the original one looked at in another way. However, to show this requires a bit of work. This too is a wonderful topic for middle school children. Closely related to this is the subject of magic triangles.

It is important to dip into Euclidean geometry to as great a depth as possible, simply because geometry is such a rich source of beautiful problems. It also happens that Euclidean geometry is a highly neglected area in school mathematics. Let me share here a few problems that have been excellent sources of class discussion.

3. Given a square *ABCD* with side midpoints *E,F,C,H*, we draw lines as in Figure 1. Find the ratio of the area of shaded region *HIFJ* to that of square *ABCD*.
   (I got this problem from the Twitter account https://twitter.com/Cshearer41.)

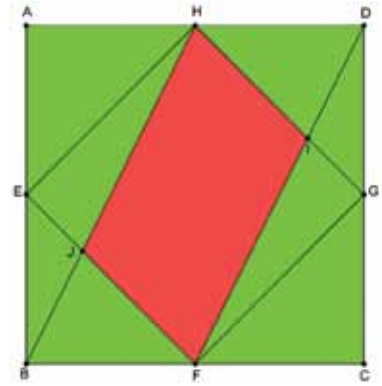

Figure 1

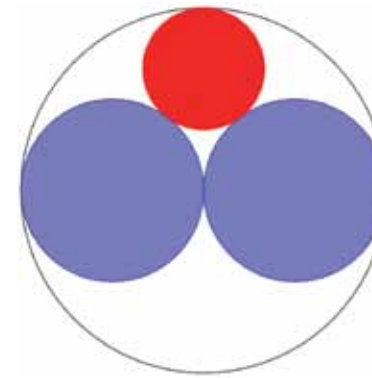
Figure 2

4. Look at the configuration of the four circles (Figure 2), all touching each other. If the largest circle has radius 2 units and the two circles shaded blue have radius 1 unit, what is the radius of the circle shaded red?

5. In Figure 3, we see a configuration of three squares in a row. Three angles *x, y, z* have been marked. Find the angle sum $x + y + z$.

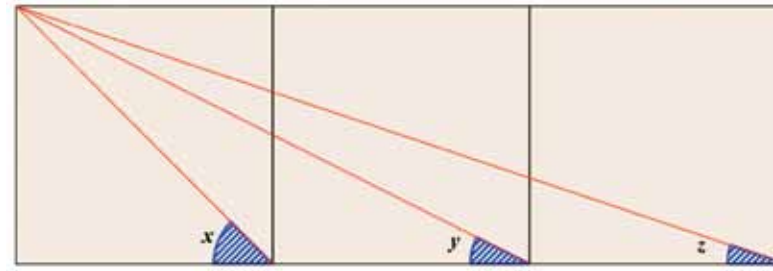

Figure 3

The last problem is very famous and it allows for some very elegant solutions.

In this short write-up, I have focused on problems at the middle school level. However, the beauty of this subject is that we can find problems that are both enjoyable and accessible to the student and the teacher alike, at every level. We can also find problems that even students and teachers who are highly capable will find challenging.

The beauty of problem solving is that it is nearly impossible to have an encounter with it that does not enrich and empower us in some way. Each time, one learns fresh the truth of Pólya's dictum, that it is better to find five different ways to solve a problem than to solve five problems in the same manner. Is it not time that we allow this dictum to come alive in our classes? Is it not time that we work towards nurturing a culture of non-routine problem solving at every level of the school? If our intention as mathematics teachers is to instill in the child a lasting love for the subject, then it is imperative for us to educate ourselves in this matter and to take forward the spirit of problem solving to as great an extent as possible.

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Developing knowticing among mathematics teachers

Shikha Takker

The author is pursuing her PhD in mathematics education from the Homi Bhabha Centre for Science Education, TIFR, Mumbai. Her doctoral research is on developing teachers' knowledge of students' mathematical thinking by supporting them in the contexts of their practice. Prior to pursuing her PhD, she worked as an elementary school mathematics teacher. She can be reached at <shikha@hbcse.tifr.res.in>.

I was observing an introductory lesson on decimals in grade 5. The teacher began by recalling the place value of the digits in the number 739. Students repeated the place value of each digit after her, "Seven hundreds, three tens and nine ones". The teacher then changed the number to 7.39, drawing students' attention to the dot and defining it as the decimal point. She added that the digits before the decimal point are a whole number. She mentioned that the place value names of the digits to the right side of the decimal point are like the whole number names, "Like tens on the left side, there is tenths on the right side, (emphasizing-ths in tenths)." She introduced tenths, hundredths and thousandths by establishing the correspondence between the place value names on either side of the decimal point. A student almost immediately asked, "Teacher, where is oneths?" There was a pause for a moment in the class after which the teacher answered, "There is no oneths." She then moved on to place the decimal number in the place value table.

In another introductory lesson on decimals, the teacher recalled length measures. Students followed her explanation of the length of a pencil as 3 centimetres 5 millimetres. The teacher gave a few more examples: 5cm 2mm, 1cm 4mm, 6cm 9mm, etc. She then told the students that 3cm 5mm can be written as 3.5 cm, where the point separates the centimetres from the millimetres. The students were expected to follow the pattern and extend this representation to the other lengths: 5.2 cm, 1.4 cm, 6.9 cm. A student then asked, "Teacher why centimetre?*" to which another student added, "Haan, why not millimetre?" The teacher moved on to explaining the place value names of the digits on the left and right side of the decimal point. She mentioned how in 56.78 cm, the place value of 5 is tens. Similarly, she added, the place value of 7 is tenths. Another student asked, "Teacher, seven should be oneths na?" The teacher answered, "The place value of 7 is tenths and not oneths."

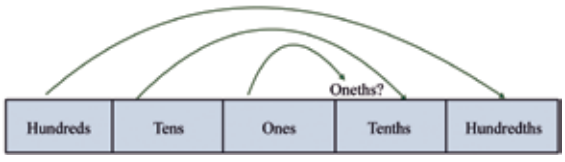

Figure 1: Where is oneths?

*What would have led the student to ask this question? How is the decimal interpretation helpful in understanding the use of this unit?

The "oneths" question raised by the students is intriguing. It seems reasonable that if there is a correspondence between the names of the whole number part (tens, hundreds, thousands) and the fractional part (tenths, hundredths, thousandths) of a number, then the same has to be true for the ones place value. The explanation of extending the names from the left side of the decimal point to the right side does not hold for a corresponding position for ones, why?

Before we understand the reason for the missing oneths, it is important to acknowledge that the question arose from the students in both the lessons. Selecting such mathematically potent ideas from a variety of students' utterances is one of the tasks entailed in the complex work of teaching. I refer to those student ideas as 'mathematically potent' which have the potential to unpack some key mathematical ideas. Such ideas can be used by a teacher as a learning opportunity for the whole class. What kind of knowledge do teachers need to deal with such ideas in the classroom?

Knowing content and students

We all agree that a teacher needs to know the content to be taught. One of the sources of this content is the textbook. However, handling students' questions arising during the course of teaching requires knowledge that is beyond the content knowledge represented in the textbooks. Additionally, handling unexpected student questions requires the knowledge of the content and the skill of dealing with it in the whole class. In the literature on mathematics teacher knowledge, such a combination of skill and knowledge is called *knowticing* (Even, 2008). Knowticing the mathematical potential of a student's idea and handling it requires an in-depth knowledge of the content. An in-depth knowledge includes knowing why an algorithm works, what kind of representations are useful in teaching a particular topic (Ball, Hill & Bass, 2005), the prior knowledge needed to learn a topic, how it connects with the topics to be taught later, and so on. A deeper knowledge of the content supports teachers' *knowticing* and vice versa.

Why is there no oneths?

Let us see why students are led to ask the question about the existence of oneths. Often, the place value names of the fractional part of the decimal number are introduced using the whole number names. The correspondence between the names of the place values is drawn around the decimal point, for instance, in references such as "left and right side of the point". Such an explanation creates a mental mirror or symmetry between the place value names on either side of the decimal point (refer Figure 1). The mirror metaphor (MacDonald, 2008) or the symmetric relation helps the students in recalling the names of the fractional parts, that is using tens to recall tenths. Teachers use the symmetric relation to bridge students' prior knowledge of the place value names learnt during whole numbers with the new knowledge of naming the fractional part of the decimal number. On the other hand, such an explanation requires locating the corresponding place value name for ones, that is, oneths.

The place value to the immediate left of the ones is 10 times ones and that to the immediate right is $1/10^{th}$ of ones. This means that the symmetry of place value names is on either sides of the ones place value and not the decimal point. In other words, the place value of the digits to the left and right side of the ones are positive and negative powers of 10, respectively. Since, the symmetry is around the ones place value, the correspondence between place value names on either side of the ones can be drawn.

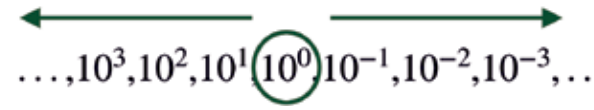

Figure 2: Symmetry around "ones"

While the mirror metaphor is one way of explaining the decimal place values, there is another explanation. In a base 10 system, each consecutive place value is related to the other by a power of 10. A continuous relation between place values of digits in a decimal number (refer Figure 2) helps in understanding why oneths cannot exist.

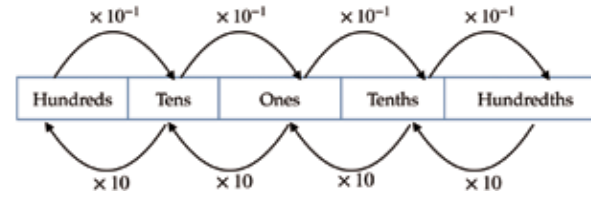

Figure 3: Consecutive place values in base 10 (adapted from MacDonald, 2008)

Both these explanations for the non-existence of oneths were discussed with the teachers who taught the two lessons reported at the beginning. As the teachers discovered the reasons underlying the students' questions, they began to look at the oneths question differently.

In the following year, the teacher began teaching decimal place value names using the continuous

relation between the place values of digits in the base 10 system. A student observed the correspondence between the place value names and asked, "Teacher, why is there no oneths?" The teacher posed this question for discussion in class, encouraging students to offer explanations. While two students argued that oneths exist, a few other students presented counter-arguments. The teacher emphasized some student utterances by posing questions, such as, (a) If oneths existed, what would it mean? (b) If oneths is not a fraction, where would its position be in a decimal number? and (c) Are ones and oneths different? The teacher together with the students discovered why oneths does not exist while emphasizing the relation between place values (refer Takker & Subramaniam, 2019 for details).

Concluding thoughts

The students' question about oneths was not anticipated by the teacher and its potential was undiscovered in the earlier lessons. When the teacher decided to discuss the question, she used the key idea of place value in a base 10 system. Sometimes as teachers, we ignore students' questions, may postpone them, or even forget them. However, we can also consider students' questions as opportunities to communicate key mathematical ideas. Resources such as textbooks, research literature on students' topic-specific conceptions#, discussions with colleagues and the students themselves are useful in handling such situations. Listening to such student ideas and responding to them in ways which are productive to developing students' mathematical thinking is challenging, yet a valuable goal of mathematics teaching.

#Probing and analying students' responses is an important way of developing such knowledge. For more details, refer to, Takker, S. (2019). Understanding learners' thinking through an analysis of errors. *At Right Angles*, Azim Premji University.

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2 MacDonald, A. (2008). 'But what about the oneths?': A Year 7 student's misconception about decimal place value. *Australian Mathematics Teacher*, 64(4), 12-15.

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Pathways to concept-mastery

Athmaraman R

The author started his career in 1962 and has worked as a teacher, headmaster, researcher, academic advisor and math educational consultant. He has served as the Secretary of The Association of Math Teachers of India (AMTI) and also as the editor of the journal Junior Mathematician. He has authored several books and written textbooks for NCERT and Tamil Nadu State Board. He can be reached at <athmaramanr@gmail.com>.

The need

To succeed in mathematics, students need competency in its content, concepts and skills. Teaching should aim to provide all three. Regrettably, the traditional objectives-based approach to teaching has not yielded the desired results. In particular, conceptual understanding is often feeble and even nonexistent among learners. For instance, many children solve multi-digit subtraction problems using the procedure of "borrow and regroup" without knowing why they do so. That the 'product of two negatives makes a positive' is an interesting result but there is little inclination among students to know why it works so. For them, to secure marks, 'how' of the procedure is more essential than 'why' of the process.

Concepts are a way to organize and make sense of learning and when they are not assimilated properly, the whole exercise of teaching and learning becomes worthless.

The belief that 'one of the most tragic shortcomings in our system of mathematics teaching is the students' lack of conceptual understanding' conceals the unpleasant fact that such a deficiency exists to some extent even among the teaching community. A mathematics teacher, who is to retire from service shortly, sent an urgent request to a consultant: "Sir, I need exact answer for the following question, which I have to discuss with my class tomorrow. They have brought this question:

Name the smallest single digit number:
A) -9 B) -1 C) 0 D) 1″

In the first place, the question is vague since it does not indicate whether it refers to a natural number or a whole number and naturally leads to uncertainty in comprehending a concept. When the same question was posed to another teacher, she responded correctly declaring that 1 is the answer if we consider the natural number system. However, she struggled a lot when she was asked to consider the question with regard to other systems, for instance, the system of complex numbers. This shows the absence of confidence stemming from the lack of awareness of the 'order relation' concept.

There is an obvious striking correlation between teachers' mastery of concepts and student achievement. Failure to grab fundamental notions leads to poor performances resulting in dejection and fear of the subject. Primary math concepts intrinsically hook up to each other, because of which the learner who comprehends one concept will figure out a consequent one easily and swiftly. If a learner is not thorough with the notion of 'a multiple of a number', how will he/she handle the idea of 'least common multiple'?

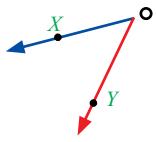
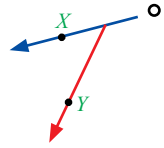
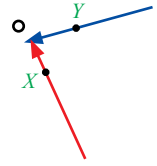
Concept mastery being the essential foundation for doing well in mathematics, the question arises as to what strategies a teacher should adopt to help the learners gain conceptual understanding. We try to list a few techniques here.

Examples and non-examples

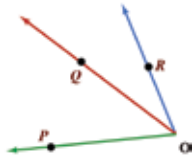
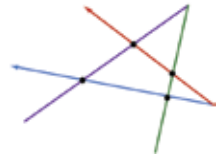
Euclidean geometry in schools is based on three essential undefined terms: a point, a line and a plane. ('Space' may be reserved for higher levels). One cannot *define* them but can quite possibly *describe* them. For example, a *dot* is used to *represent* a point. (A dot has a little breadth and height and so is not a point by itself but only a representation!). A point tells where something precisely is and begins to build geometry.

A point is an exact location	A line is an assembly of points continuing forever in both directions. A line is always 'straight' and there is no 'bend'.	A line segment is a portion of a line with two end-points.	A ray starts from one end point (named vertex) and extends in one direction forever.

One can now *define* an angle. An angle is a pair of rays (called sides) with a common end point (called vertex). An angle is a very important concept, which is often not understood entirely.

		
This shows an angle. Written $\angle XOY$ *or* $\angle YOX$, Named as angle XOY or YOX	Is there an angle here? Is it angle *XOY*?	Is there an angle here? Is it angle *XOY*?

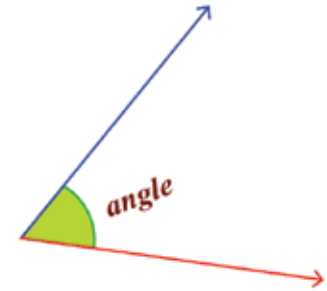
To strengthen the concept further, discussion on questions of the type that follow may help the teacher:

	
In the figure, is *PQR* an angle? How many angles do you find here? Name them.	Here are two angles meeting in 4 points. Can we draw a different figure for the same condition? Can we draw two angles which intersect in 3 points? 2 points? 1 point? No point? 5 points?

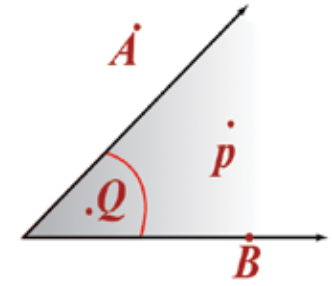
Exploring different possibilities, thus leading to a lot of communication between the teacher and the taught, will establish clarity in thinking.

Quite often, one finds figures like the ones given here:

This figure may lead to an erroneous idea that the green-coloured *area* is the angle, whereas in reality, the angle is simply the couple of rays and has nothing to do with the green-coloured zone. What if a student draws a larger green zone for the same angle? Will it be a different angle? If the angle is viewed through a magnifying glass, the shaded area will seem bigger and what will it mean then? One may feel that this is a silly difference to point out, but we will see how it is important.



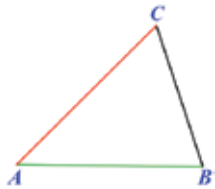
In the figure shown here, which are the points that belong to the angle? Only *B* is the eligible candidate. Just carefully go through the definition of an angle above. P and Q are in the *interior* of the angle while A is in its *exterior*. The learner has to be given plenty of illustrations as examples and non-examples.



[To digress:

1. An angle may also be defined as the amount of 'turn' between two rays around their common starting point; in this case the concepts have to be listed and approached appropriately in a different way.

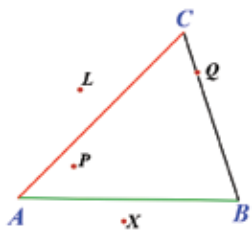
2. Here we have not dealt with the very important difference between an *angle* and the *measure of an angle*. The practice of marking with an arc the interior of an angle perhaps is due to the mix-up between these two concepts].



How do you define a triangle? It is a closed figure consisting of three line segments linked end-to-end. While giving such a definition, the teacher should keep in mind that it is linked to another concept, "closed figure". It also assumes that the three points where the line segments meet are not "collinear" (which is one more concept, left usually to the intuition of the learners). Because of this difficulty, a strict definition would be as follows:

Given three points A, B and C (called *vertices*) that do not belong to the same line, the union of segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ (called *sides*) is called a triangle (denoted by ΔABC). Note that the sides can be seen as parts of rays and so $\angle BAC$, $\angle ABC$, and $\angle ACB$ are the angles of the triangle.

A point lies in the interior of a triangle if it lies in the interior of each of the angles of the triangle. A point lies in the exterior of a triangle if it lies in the plane of the triangle but does not lie on the triangle or in the interior.

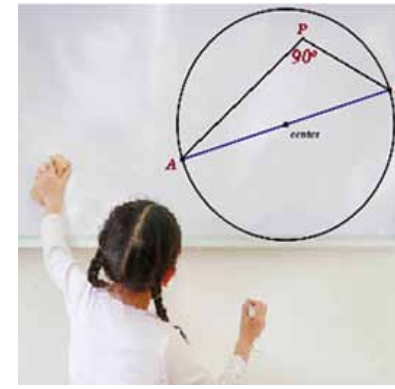


In the figure, does *P* belong to the triangle? No, it is in its interior. How about Q? Surely it belongs to the triangle since it lies on the segment connecting *B* and *C*. The points *X* and *L* are in the exterior of the triangle.

The teacher should be able to create adequate non-examples and examples.

Study the following situation

The teacher has explained the theorem about the angle ***in*** a semicircle.



The student draws a figure and asks if it is correct since P is ***"in"*** the semicircle.

This confusion is due to the misconception that the *interior* also is part of the circle. The circle, by definition, is made up of the curve only. The point *P* has to be a point *of* the curve for the theorem to be true.

Many such examples can be given where such false impressions arise, pointing to the need to use illustrations and counter-illustrations to make things unambiguous. This will help to gain a precise and accurate understanding of the concept. Learners need to be told how our 'loose' (viz, non-rigorous) communications actually mean precise articulations.

For example

Imprecise statement	*Precise statement*
The *sum of* the three *angles* of a triangle is 180°	The *sum of* the *measures* of the angles of a triangle is 180°
The perimeter of a square is the *sum of* its *sides*.	The perimeter of a square is the *sum of* the *lengths of* its *sides*.

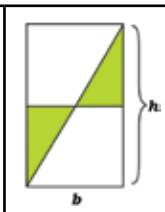
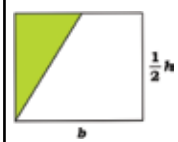
While aiming at precise conceptual understanding in teaching, teachers should be able to give a number of precise and imprecise definitions in contrast.

Visualization

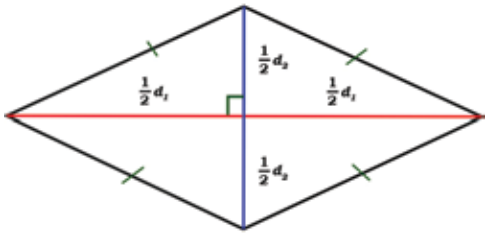
Visualization in mathematical learning can achieve a lot. It entails setting up a link between information about concepts, which are previously unknown, and understanding, which gradually develops. It leads to the creation of mental images of given concepts. At the beginning stage, visualizing a fraction as a part of a whole is a familiar example. It is the experience of Singapore mathematics that the sequence "abstract ⇒ visual model ⇒ symbol" influences learners to do problems in their heads, to articulate them out and to draw visual images of the problem before using the concepts and language of mathematics. In

fact, algebraic ideas are disseminated in a concealed manner. Visual justification of concepts involved is the preliminary outcome, which can be strengthened by rigorous supplements subsequently. While one need not and cannot depend on this technique always, the capacity to visualize can often simplify the process. Encouraging the learner to visualize will be quite rewarding. Diversity in approaches can sometimes be an added bonus.

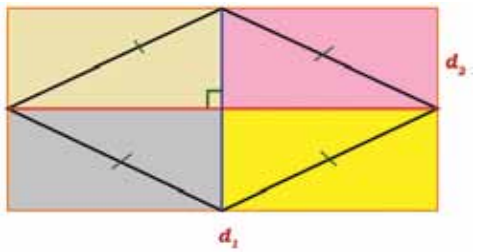
Observe how two learners may approach the problem of finding the area of a given right triangle:

Learner 1		The given triangle is half of the rectangle	
Learner 2		The shaded area is a dissected portion of a rectangle.	

Similarly, while computing the area of a rhombus, one may visualize it as a union of four isosceles right triangles or as half of a rectangle passing through its vertices.

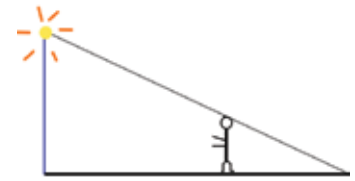


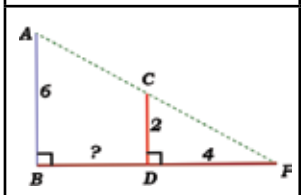
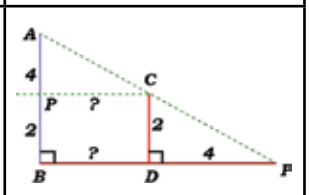
Or



Visual reasoning is not restricted to geometry or spatially represented mathematics. Here follows an example based on the concept of proportions.

A boy of height 2 metres stands at a certain distance from a lamp post of height 6 metres and casts a shadow of 4 metres on the level ground. Find the distance between the boy and the lamp post.



Learner 1	Learner 2
	
AB – Lamp; CD – Child; DP – Shadow Let h be the height of the lamppost. Since $\triangle ABP$ and $\triangle CDP$ are similar, $\frac{AB}{CD} = \frac{BP}{DP}$ $\therefore \frac{6}{2} = \frac{x+4}{4}$ or $x+4 = 4\times 3$ $= 12$ $x = 8m$	Visualize a line CP through C parallel to PB. AP is 2 times CD. $\therefore$ PC and hence PD must be twice DP. Thus DP = 8 *m*.

For learner 2, visualization shortens the calculations, since clarity of concept of proportions seems to influence him via his mental images.

Let us consider one more example, now from arithmetic.

Mr Bumble withdrew some money from a bank. He used half the amount to buy things from a supermarket. He spent one-third of the remaining sum towards the purchase of some books. Lastly, he spent one-fourth of the sum left over to pay mobile phone charges. Now he had a balance of Rs.900 with him. How much did he withdraw from the bank?

Learner 1:

Let the amount drawn from the bank	$= Rs.\ x$
Expenses at the supermarket	$= Rs.\ \frac{1}{2}x$
Balance	$= Rs.\ x-\frac{1}{2}x=\frac{1}{2}x$
Cost of books bought	$= Rs.\ \frac{1}{3}\times\frac{1}{2}x=\frac{1}{6}x$
Balance	$= Rs.\ \frac{1}{2}x-\frac{1}{6}x$
	$= Rs.\ \frac{(3-1)}{6}x=\frac{1}{3}x$
The mobile phone charges	$= Rs.\ \frac{1}{4}\times\frac{1}{3}x = Rs.\frac{1}{12}x$
Final balance	$= Rs.\frac{1}{3}x-\frac{1}{12}x$
	$= Rs.\frac{(4-1)}{12}X$
	$= Rs.\frac{3}{12}x = Rs.$
If Rs.$\frac{1}{4}x$ is final balance, amount drawn	$= Rs.x$
If Rs.600 "	$= Rs.\frac{x}{(\frac{1}{4}x)}\times 900$
	$= Rs.3600$

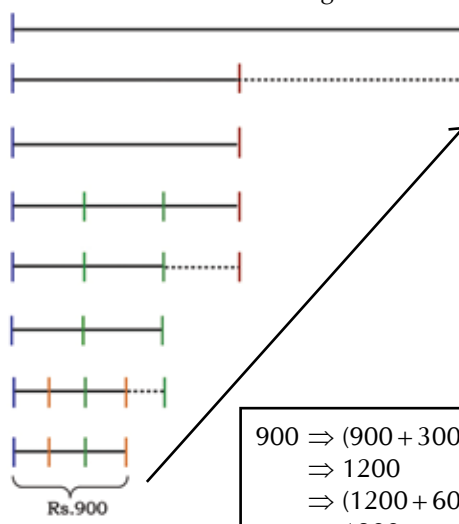
Learner 2:

Let the amount drawn from the bank = $Rs.100$
Expenses at the supermarket = $Rs.\ 50$
Balance = $Rs.\ 50$
Cost of books bought = $Rs.\ \frac{1}{3}\times 50=\frac{50}{3}$
Balance = $Rs.\ 50-\frac{50}{3}$
= $Rs.\ \frac{150-50}{3}=Rs.\frac{100}{3}$
The mobile phone charges = $Rs.\ \frac{1}{4}\times\frac{100}{3}=Rs.\frac{100}{12}$
Final balance = $Rs.\frac{100}{3}-\frac{100}{12}$
= $Rs.\frac{400-100}{12}$
= $Rs.\frac{300}{12}=Rs.25$
If Rs.25 is final balance, amount drawn = $Rs.100$
If Rs.600 " = $Rs.\frac{100}{25}\times 900$
= $Rs.3600$

Learner 3:

Steps — Mental Image

1. This is amount drawn
2. Half at supermarket
3. Balance
4. Divide it into 3 parts
5. One-third for books
6. Balance
7. Divide it into 3 parts
8. Last balance.



$900 \Rightarrow (900+300)$
$\Rightarrow 1200$
$\Rightarrow (1200+600)$
$\Rightarrow 1800$
$\Rightarrow 1800$
$\Rightarrow (1800+1800)$
$\Rightarrow 3600.$

The image sequence in reverse gives (moving from step 8 towards step 1)

The first two learners follow the traditional approach whereas the 3rd learner has conceptualized the happenings with the help of a visualized mental image, which has given an elegant way of solving the problem.

We note that the process of visualization could be through pictorial images or schematic representation. It is crucial that the teacher makes this clear to the learners. An experienced teacher will also find that most students gifted in mathematics are non-vizualizers.

It is perilous to always take visual justifications as 'proofs'; visual thinking in mathematics fell into disrepute since it can often be deceptive' (a justification of this statement is beyond the scope of this article). Nevertheless, after computer science began to boom, visual thinking rehabilitated the epistemology of mathematics.

Probing questioning

To promote inquiry into and perception of various aspects of concepts, Socratic dialogue between learners and the teacher is crucial, based on asking and answering a series of interlocked, sequential questions, to stimulate critical thinking and to draw out ideas. Asking questions at the end of a lesson as a ritual does not serve the purpose of questioning. Very few teachers are aware of the distinction between formative evaluation and summative evaluation and hence do not provide for punctuating teaching with questions that constitute formative evaluation – that is, checking whether the learner has mastered every concept before the teacher goes to the next concept/point.

Questioning should be utilized to clarify notions, challenge assumptions, examine reasons and perspectives and to explore implications and outcomes of proposals and statements made during discussion. Frequently, the responses may be inappropriate-off at a tangent, invalid and meaningless, but the learner should be allowed to make mistakes. The best way to learn concepts is to learn how the 'trial and error' mode of learning will make for a stronger grasp of concepts.

Here are some samples that direct towards clarification of concepts:

1) *Does 'zero' mean 'nothing'?* (If there is zero degree temperature on a hill station, does it mean there is no temperature?

Write the numeral 'twenty three'. You get 23. What is between 2 and 3 here? Nothing! Then why don't you put the numeral for 'zero' there?)

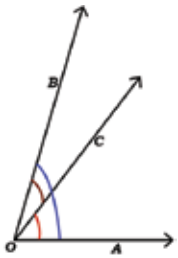
2) *It is said 0.9999…. is equal to 1. When you have endless number of 9s following one after another, how is this possible?*

Method 1 Method 2 Method 3

3) *What is the value of the product* $0^5 x 0^{-5}$*?*

($0^{-5} = \frac{1}{0^5}$ is undefined)

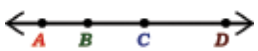
4) *Comment: "Adjacent angles are two angles that have a common vertex and a common side."*
(Study the figure shown here. ∠AOC and ∠BOC are adjacent angles. Are ∠AOC and ∠AOB adjacent? No doubt, they have a common vertex and a common side. Adjacent angles should not have common interior points!)



5) *Comment: "The slope of a line is zero. Therefore it has no slope."*
(A horizontal line has a slope equal to zero; a vertical line will have no slope, and "the slope is undefined".)

6) *Comment: "If the measures of the corresponding angles of two polygons are equal, their sides will be proportional."*
(No. Try to draw two rectangles whose sides are not proportional.)

7) $\overleftrightarrow{AC}$ is one way of naming the line shown here. In how many more ways can it be named?
(11 more ways.)



8) *Why cannot 48 be the LCM of 18 and 24?*
(48 is not a multiple of 18; 18 does not divide 48 evenly.)

9) *One side of a right triangle is 5 cm. Find the lengths of the other two sides.*
(The students may be tempted to give the answer as 3 cm and 4 cm, thinking that only the hypotenuse will have length 5 cm. There are many answers!)

The last two questions belong to the category of *open-ended* questions. These are quite useful to initiate a forceful debate of the concept discussed.

Investigations and project work involving collaborative learning are some of the other avenues available to promote conceptual mastery.

(Open-ended questioning, investigations and project work are themselves topics deserving separate discussion and are not a part of this article).

Conclusion

To enable concept-mastery, teachers need to encourage divergent and reflective thinking. Coupled with appropriate questions, interactive sessions would promote confidence and participation that induce healthy competition.

Contributions made by students make it an enjoyable experience for them. Enabling visualization, providing illustrations and counter-examples and facilitating a communicative environment in the classroom are *some* ways to achieve concept mastery.

TP

THE ROPE IS 29 mts DIVIDE IT AMONG 4 PEOPLE
29 is a Prime Number. 29÷4 to be done in a Calculator
I would fold the Rope in half and again half and cut
MATH TEACHER
STUDENT
VILLAGER

Investigating irrationals

R. Sivaraman

The author is Associate Professor of Mathematics, D.G. Vaishnav College, Chennai. He has more than two decades of teaching experience. He has provided more than 300 lectures conveying the beauty and application of mathematics. He has written 20 books to popularize mathematics among the common man. He has won more than 30 awards for his distinguished service to mathematics. He can be reached at <rsivaraman1729@yahoo.co.in>.

It is well-known that real numbers are comprised of both rational and irrational numbers. Though it is fairly easy to understand rational numbers (which are the quotient of two integers of the form say p/q where q is non-zero and p, q are relatively prime), the understanding of irrational numbers has been quite a nightmare for students as well as many others. The most interesting aspect of irrationals is that they are present throughout the number line and yet it is quite difficult to recognize them as precisely as their counterparts, the rational numbers.

History tells us that $\sqrt{2}$ is the first discovered irrational number. In particular, a disciple of Pythagoras, upon investigating numbers suddenly stumbled upon it, when he tried to determine the diagonal of an isosceles right triangle whose legs are of one unit length each.

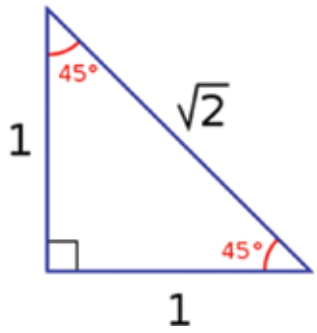


Figure 1

This discovery led to the complete devastation of Pythagorean principles, which believed that everything in the universe is expressible through rational numbers. But here was a line segment of some particular length which was not equal to any of the known rational numbers. This became the first step in our understanding of irrational numbers. Often, one of the very first exercises for training in mathematics begins with the classic theorem proving that $\sqrt{2}$ is not rational. Here we see some versions of proving $\sqrt{2}$ as irrational. Using this as the template, we can prove that several classes of numbers are not rational numbers.

Proving irrationality

We begin our exploration by first showing that $\sqrt{2}$ is not rational.

Theorem 1

The number $\sqrt{2}$ is not rational.

First proof

We present the standard classical proof, which is quite well-known. We follow a method of proof called "Proof by Contradiction", in which we assume the conclusion to be false and prove eventually that the hypothesis itself is false. Hence, if the hypothesis is true, then the conclusion must be true.

So, we start by assuming that $\sqrt{2}$ is a rational number say of the form $\sqrt{2} = \frac{p}{q}$ where *p, q* are relatively prime and *q* is the smallest possible denominator (that is, it is in reduced form).

Now from $\sqrt{2} = \frac{p}{q}$ we get,

$2p^2 = q^2$. Since q^2 is even, *q* must be even. Otherwise, if q is odd, then from the equation $2p^2 = q^2$, we find that the left hand side is even, whereas the right hand side is odd, which is not possible. Hence, *q* must be even, say $q = 2s$ for some integer *s*. With this value of *q*, we have $2p^2 = q^2 = 4s^2 \rightarrow p^2 = 2s^2$. As above, p^2 is even and so *p* must be even.

But wait, we have come to a situation where both *p* and *q* are even, contradicting the assumption that *p* and *q* are relatively prime. Moreover, since *p* is even, we have $p = 2r$ for some integer *r*. Also note that $r < p$, $s < q$. Hence from $\sqrt{2} = \frac{p}{q}$ we have $\sqrt{2} = \frac{p}{q} = \frac{2r}{2s} = \frac{r}{s} \rightarrow \sqrt{2} = \frac{r}{s}$.

Note that we have written $\sqrt{2}$ as $\frac{r}{s}$ where $r < p$, $s < q$ contradicting the assumption that *q* is the smallest possible denominator in expressing $\sqrt{2}$ in the simplest rational form. Hence, these contradictions prove that $\sqrt{2}$ is not rational and so must be irrational.

Second proof

We now present the second proof by American logicist Stanley Tennenbaum. This proof will provide a visual representation of the required truth. As with the first proof, we assume that $\sqrt{2}$ is rational say $\sqrt{2} = \frac{a}{b}$ where *a, b* are integers. Since $\sqrt{2} > 1$, we see that $a > b$ and *b* is the least possible denominator such that $\sqrt{2} = \frac{a}{b}$.

Now from $\sqrt{2} = \frac{a}{b}$ we get $a^2 = 2b^2$. We try to construct a square of side length a as shown in Figure 2.

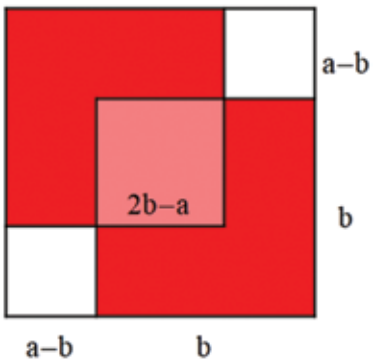


Figure 2

We see that in the whole square of side length a, there are two red squares of side length *b* which overlap forming a pink square of side length $a - (a-b) - (a-b) = 2b-a$ as shown in Figure 2.

Since $a^2 = 2b^2$, the area of the two red squares of side length *b*, must be equal to the whole square whose side length is *a*. But according to Figure 2, we see that two red squares miss two smaller squares of side length a-b, but double count a square with side length $2b-a$.

Consequently, the double counted region must have the same area as the two missing squares, that is, $(2b-a)^2 = 2(a-b)^2$ giving $\sqrt{2} = \frac{2b-a}{a-b}$. Also we observe that since $a > b$, $2b-a < a$. Similarly, since $2b-a > 0$, $a-b < b$. But this contradicts the fact that *b* is the smallest possible denominator in expressing $\sqrt{2}$ as a rational number. This contradiction thus proves that $\sqrt{2}$ is not rational.

We arrived at the same conclusion as in the first proof but this time presented it in visual form. There are at least half a dozen other proofs that can be adopted in showing that $\sqrt{2}$ is irrational. We now prove a more general theorem regarding identifying irrationals in square root radical forms.

Theorem 2

If a natural number *n* is not a perfect square, then $\sqrt{n}$ is not rational.

Proof

We adopt the same technique of Proof by Contradiction to show that $\sqrt{n}$ is irrational.

If possible, let $\sqrt{n}$ be rational of the form $\sqrt{n} = \frac{a}{b}$, where *a, b* are relatively prime integers such that *b* is the smallest possible denominator in expressing $\sqrt{n}$ in such form.

Since *a, b* are relatively prime integers there exists integers *x, y* such that $ax + by = 1$. Now from $\sqrt{n} = \frac{a}{b}$ we have $a = \sqrt{n}b, \sqrt{n}a = bn$.

We now make the following manipulation:

$$\sqrt{n} = \sqrt{n}x1 = \sqrt{n}x(ax + by) =$$

$$(\sqrt{n}a)x + (\sqrt{n}b)y = bnx + ay$$

Since, *a, b, n, x, y* are all integers it follows that $bnx + ay$ is also an integer. Hence $\sqrt{n}$ must also be an integer implying that *n* is a perfect square, contrary to our assumption that *n* is not a perfect square. This contradiction proves that $\sqrt{n}$ is not rational.

Note:
Theorem 2 helps us generate infinitely many irrational numbers each of square root radical type.

We now prove a more general theorem regarding irrational numbers, but before doing that we provide the following known theorem (without proof) famously called as Fermat's Last Theorem. This notorious theorem was proved by English Mathematician Andrew Wiles in 1995.

Theorem 3: (Fermat's Last Theorem)
There do not exist natural numbers a, b, c, n such that $a^n + b^n = c^n$ if $n \geq 3$.

However, in the words of the great French mathematician Fermat, who is the originator of this theorem, "Mathematicians have discovered truly marvelous proof of this, which this magazine is too narrow to contain".

Using Fermat's Last Theorem as stated in Theorem 3, we now prove the following theorem.

Theorem 4
The number $\sqrt[n]{2}$ is not rational for any natural number n, such that $n \geq 2$.

Proof

If $n = 2$, then $\sqrt[n]{2} = \sqrt{2}$ is not rational by Theorem 1. Hence, we assume $n \geq 3$. As in the previous theorems, let us assume that $\sqrt[n]{2}$ is rational and arrive at a contradiction.

Let $\sqrt[n]{2} = \frac{a}{b}$, where a, b are some natural numbers. Then $a^n = 2b^n = b^n + b^n$. Thus if $\sqrt[n]{2}$ is a rational number, then the sum of two n^{th} powers is another n^{th} power, contradicting Fermat's Last Theorem stated in Theorem 3. This contradiction proves that $\sqrt[n]{2}$ is not rational.

Conclusion
In this article we not only proved that $\sqrt{2}$ is irrational but several class of numbers of the form $\sqrt{n}$ (n is not a perfect square) and $\sqrt[n]{2}$ are irrationals using the same technique of Proof by Contradiction. This method is the finest weapon that is used by mathematicians to prove numerous theorems in most branches of mathematics.

TP

LOOK DEAR HOW MATH INTELLIGENT THEY ARE! THEY DON'T WANT 4 IDLIS. THEY WANT ONE FIVESTAR AND ICECREAM. THEY SAY BOTH ARE EQUAL IN CALORIES
5 STAR
ICE CREAM

The role of technology in math education

Kannan Bharadwaj

The author is the Co-Founder & CEO of Math Buddy (www.mathbuddyonline.com), a company focused on providing tools to help teachers teach concepts and for students to learn with understanding. He can be reached at <kannan@mathbuddyonline.com>.

Technology has gained a new prominence in the wake of the COVID-19 pandemic. It is not just office goers who are working from home but students are learning from home as well. In these highly uncertain times, schools are exploring digital ways of bringing learning to students in their homes. Technology is slowly transforming from being a supplemental resource to becoming an integral part of education. This article examines some advantages of technology, especially in math education, and the changes required in the school ecosystem to ensure that it is impactful.

Math is a subject that is best learnt through exploration. It is proven that hands-on exploratory learning is very effective in helping children grasp concepts, especially at the primary level. Towards this end, most math textbooks have changed to include more activities, thereby making math learning interesting and easy to relate to everyday life.

Unfortunately, most schools teach mathematics in a procedural way. We teach students ways and methods to find solutions to problems and students just follow the steps as outlined by the teacher to arrive at the answers. There is no clarity on what exactly happens when we follow these steps. To a large extent, hands-on activities help students explore and find out what exactly happens when we follow those steps. With hands-on instruments, concepts such as the number system, fractions, decimals, operations and geometry can be modelled and internalized with so much ease. Even in higher classes, integers and algebra including polynomial operations, factorization, etc., can be understood a lot more easily with hands-on instruments such as algebra tiles.

However, when we get to the higher classes, there are so many concepts and theorems that students just learn by rote due to lack of tools to help them visualize and understand. Here is where technology can help teachers illustrate concepts in the classroom and students learn with understanding. With the advances in technology over the last two decades, there is so much that can be done in leveraging technology to make math teaching and learning more effective.

Learning with understanding

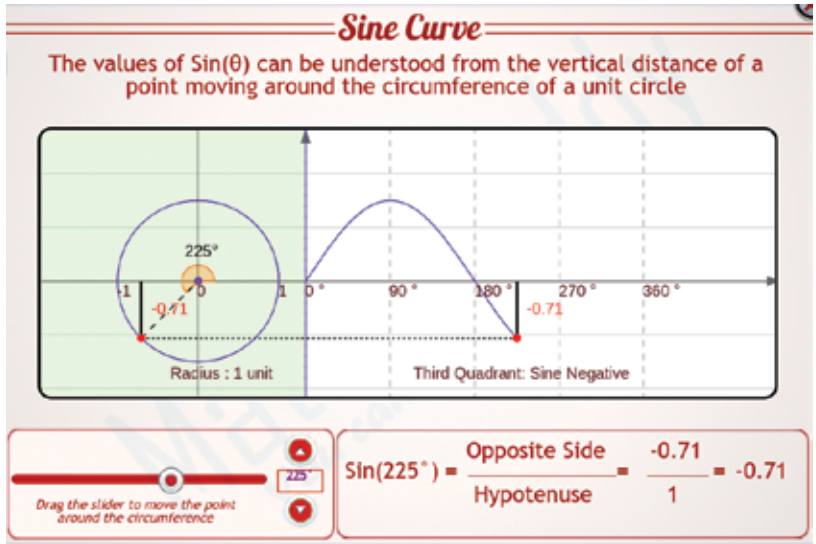
Technology can bring abstract math concepts to life, be it through visual illustrations or through videos. Every theorem in geometry can be visualized and understood better with the help of technology. Students do not need to learn by rote why the sine of an angle is positive in the first and second quadrants and negative in the third and fourth quadrants. They can actually see it in action and understand why this is the case. There are freely available tools such as GeoGebra that allow a teacher to bring many of these concepts to life with a little bit of effort. There are also paid tools like Math Buddy which provide curriculum aligned ready-to-use simulations that the teacher can use to help children visualize and understand math concepts.

Even in lower classes, while hands-on learning is very powerful in understanding concepts, technology can play an important role in reinforcing concepts. Bringing in a CPA approach (Concrete, Pictorial, Abstract) into our instruction can provide lots of benefits since we are able to cater to all types of learning styles – visual, auditory and kinesthetic.

to provide personalized learning. In a typical classroom of 30-40 students, it is very difficult for teachers to provide personalized attention to every student. This, combined with the pressure to finish the curriculum, pushes teachers to take a 'one size fits all' approach in their instructional strategies. Technology tools can help assess individual student needs and provide feedback to the teachers on how much the students have understood the lecture, which can help teachers tremendously to improve their instructional strategies.

Technology has also enabled many educators to adopt alternative instructional strategies such as flipped classrooms which can be invaluable in moving the focus away from "teaching" to "learning". The learning happens outside of the classrooms and a typical classroom session is transformed into more of a discussion session between the students and the teacher. There are also benefits of peer learning with the teachers playing the role of facilitators in the learning process.

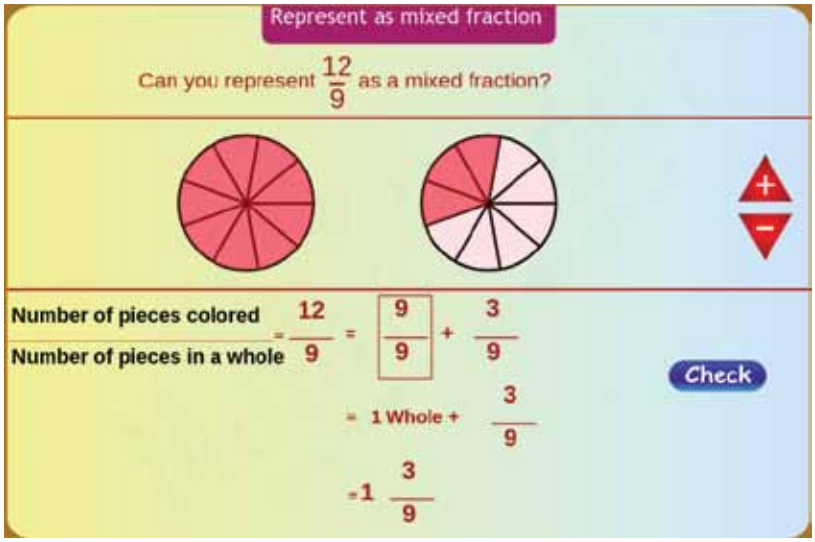
Math is one subject, where in addition to understanding, practice plays a key role in helping students master problem solving skills. Technology can be harnessed in very productive ways to help students practice solving various types of problems customized to each student's needs. The programs can adapt to the level of the student and help them move up mastering increasingly difficult questions. In addition, technology can help in keeping this practice interesting and engaging for students through gamification. There are various resources available that tap into children's interest in playing games on their computer or mobile devices by gamifying learning environments.


Sine Curve
The values of Sin(θ) can be understood from the vertical distance of a point moving around the circumference of a unit circle
225°
-0.71
0°
90°
180°
270°
360°
-0.71
Radius : 1 unit
Third Quadrant: Sine Negative
Drag the slider to move the point around the circumference
225°
Sin(225°) = Opposite Side / Hypotenuse = -0.71 / 1 = -0.71

Communication and collaboration

While the tools described above are content-specific tools which help educators deal specifically with subject related content, there are also numerous content neutral tools that play a significant role in education in the times of a pandemic. These are primarily communication and collaboration tools such as Google Classroom and WhatsApp, which are slowly becoming an integral part of many schools. In fact, tools such as WhatsApp have played a stellar role in educators extending education beyond the four walls of a classroom and being available for students anytime, anywhere.

With the availability of smartphones in every household, data being available in plenty at almost no cost and vast improvements in streaming tools, teachers can teach from anywhere and impact not only the students in their classrooms, but millions of students in any corner of the world. Students do not have to fret over not understanding a particular


Represent as mixed fraction
Can you represent 12/9 as a mixed fraction?
Number of pieces colored
Number of pieces in a whole
12/9 = 9/9 + 3/9
Check
= 1 Whole + 3/9
= 1 3/9

concept, as a simple Google search will throw up hundreds of resources from various teachers. In fact, free resources such as Khan Academy provide students with high quality visual explanations of many concepts and have become a great leveller in making education available and accessible to every student across the world.

Another important role technology can play is to enable teachers to measure students' level of understanding in the classroom in a non-threatening way. For example, technology devices such as Classroom Clickers are used extensively in some countries where students answer questions without the fear of being shamed and teachers get a clear idea of the level of understanding of the students. Technology also can provide an environment conducive to discussions and collaboration, which is essential for understanding abstract concepts.

Teacher training and continuous learning

While it is wonderful to have so many resources for students beyond the classrooms to help them understand concepts, it also means that teachers have to keep learning to ensure that they are up to date with various techniques to teach a particular topic. With teachers being responsible for many other duties in a typical school other than teaching, most find it difficult to find time to keep themselves updated and relevant. This can cause a great deal of harm in the long run where students begin to think they know better and start disregarding the teacher.

Teachers must earn to accept how technology can enhance students' learning and take steps to integrate technology into their day-to-day teaching. Schools must also take steps to not only organize many sessions with experts who can guide the teachers, but also ensure that teachers have access to the necessary hardware with software applications and web resources. There needs to be a lot of investment in training teachers in the usage of technology tools and helping them get comfortable enough to start using them in their classrooms.

Therefore, it is imperative that schools set aside time for teachers to learn, be it keeping themselves up to date with technology or sound pedagogical techniques to help students learn better. There needs to be a lot of emphasis on teacher education and professional development to continually update teachers' technological capabilities and resources to support teaching and learning. In addition to training in use of technology, teachers need to be trained continuously in their specific subjects and how they can take advantage of technology and integrate digital tools in daily instruction.

Education policy changes

Lastly, the advantages of technology described above can be derived only from systemic changes in our education system. One of the most important changes required is to move away from testing the students' rote learning capability and start testing their concept understanding. Our curriculum needs to emphasize activities and exploration which will help students get a clear understanding of what they are learning. We need to change from a system where we spoon feed all the information to one which encourages them to explore and learn things on their own. The focus has to change from "teaching" to "learning" where the teachers play the role of facilitators and help students in their path of discovery and technology can play a stellar role in making this happen.

TP

Computer programming – a road to mathematics

Sanjeev Ranganathan

The author is the recipient of the Lewis Winner award and has built over a billion products as an engineer. He moved to Auroville seven years ago and started working with rural children and youth. He set up C3STREAM Land, a centre where children come to learn mathematics, programming, electronics, leadership and much more. He can be reached at <sanjeev.ranganathan@gmail.com>. To know more about his learning centre and its work visit www.auraauro.com.

Forty years ago Seymour Papert (mathematician and a pioneer of programming) commented, "Is the child programming the computer or is the computer being used to program the child?" Technology in education has come a long way since then. In this article I look at how children learn (in this case, mathematics) when they use computer programs.

The case studies cited here mostly use Scratch (a programming language for children) and are based on the work at C3STREAM Land (Science, Technology, Research, Engineering, Arts, Mathematics), rural resource centers around Auroville, a universal township dedicated to human unity. The children coming to these centers from Auroville and surrounding villages are provided access to puzzles, games, programming, robotics and various mathematical materials.

As a pattern identifier

Children can program the computer to simulate a thought pattern and easily spot patterns they couldn't before.

Pooja (8th grade) is learning cubes and cube roots; she often mistakes x^3 (x multiplied by itself three times) with 3x (x multiplied by three). In an attempt to help her deal with and understand large numbers I suggest that she can guess the cube roots of large perfect cubes (e.g., 830,584) by estimating how big a number is (how many 1000s) and looking at its unit digit. I hope this will help her get a sense of numbers. But she is unable to follow the procedure and finds the going tough (she is not alone in this).

She then starts developing a program to find the cube roots of numbers. Her initial goal is to print the first 10 cubes. Power is not an available expression and she needs to construct the expression for a number (variable). In time

Photos courtesy: Sanjeev Ranganathan

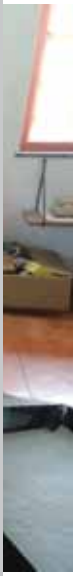
she creates result = number × number × number. She increases the number each time and puts it in a loop. To view the results, I ask her to put in a delay of 1 second after each calculation. I ask her to notice the numbers, but she still doesn't find the pattern.

She then changes the loop condition to keep it running till the result is a large cube given in her book. She now becomes interested in where the program stops and intently looks at the results. She soon figures out when she is too far and needs to wait for the result and asks to reduce the time between calculations.

I tell her she could change the program and have fewer calculations, but retain the time after a calculation to see the result. She decides to skip numbers in her program. I connect this to the original process and ask her to change in steps of 10 (10, 20, 30, etc.) till the result is too large, go to the previous step and then go in steps of 1.

She implements this as two loops: first loop in steps of 10 and the second in steps of 1. It takes her time and she makes errors, but she understands what she is trying to do and debugs it with known cube numbers and their cube roots. To use the program she generates a random two-digit number (a feature available in Scratch) and uses its cube as the target (variable). This time when she looks at the steps of 10 she notices that the last three digits are 0s. She then notices that the non-zero digits have the same pattern as cubes of single-digit numbers. When the second loop starts, the numbers are much more complicated. I ask her to focus on the units place. Now, she notices a pattern-1 in the units gives 1 in the units of the cube (21^3 = 9261). Similarly, 4 gives 4, 5 gives 5, 6 gives 6, 9 gives 9. For the other digits between 1 and 9, she notices that 3 flips with 7 (43^3 = 79507 and 57^3 =185193) and 2 with 8. She starts working this out systematically as the computer does and now she gets it. In time she skips the 'unnecessary' steps and gets straight to the cube of the 10s and then writes the number including the units. She makes the program a game that accepts inputs to check her answer. The program still works through all the steps for her to crosscheck her thought process and then announces the result. In the next class she works out 50 cube roots in an hour in her notebook and checks with the computer. She gets one wrong and understands why that one confused her.

Not only does Pooja use the computer to identify a pattern she was unable to see, she also uses it to get a sense of numbers, thinking logically while learning a procedural skill and developing an understanding of the concept. We also notice that in the process of creation, the learning becomes her own. This is the aspect emphasized in the constructionist paradigm.



As a way to look at scaled down problems

A few children from 10th grade design a physical game of Venn diagrams for three sets. Each circle in the Venn diagram represents a set constructed by a rule. There are three rules – one for colour, one for shape and one for size. In each rule there are three choices. The rule for colour can be red, green or blue.

The rule for shape can be triangle, square or circle. The rule for size can be small, medium or large. The goal is to guess the rules of each set in a minimum number of chances.

This physical game inspired Diva, a 9th, grader to make an interactive game in Scratch with the computer choosing a rule randomly. This required Diva to break the big game that was too complex into a smaller game with two sets. Diva also realized that in order for the computer to understand which region of the Venn diagram was being accessed, he needed to divide the Venn diagram (for 2 circles) into A-B (in A and not in B), B-A, A∩B (in A and B), and U-AUB (outside the two circles).

Even though visually it looked like a single picture it was in fact composed of four different pictures (or sprites). This also helped him understand the different regions of the Venn diagram better. A snapshot of the game in action is shown in Figure 1.

Diva first made the game with fixed rules which did not interest him much so he wrote out a program that allowed the computer to randomly pick the rules. This made the game challenging for him too.

Creating games like these not only help children understand concepts better, but they also push children to learn more. Just as Diva had been inspired by a game made by others, 8th graders were inspired by Diva's game, wanted to understand how it works and learn about sets even though they did not have it in their syllabus yet. At C3STREAM Land we have sessions where children share their projects once a week to encourage such learning.

Unleashing creativity using art for expression

The children who come to C3STREAM Land make many projects to demonstrate their learning and use these as presentations for other children to learn concepts. Some of these are remarkably detailed and artistic. A child working on coordinate geometry decided to create an interactive game that allowed the user to enter points one-by-one to form a picture, which when completed became a van as shown in Figure 2.

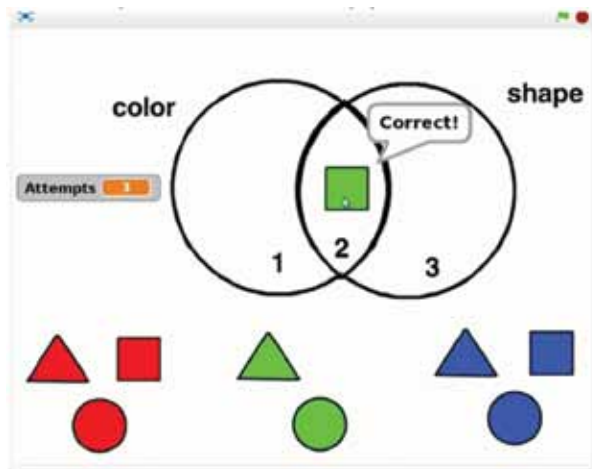


Figure 1: Interactive scratch project made on sets.

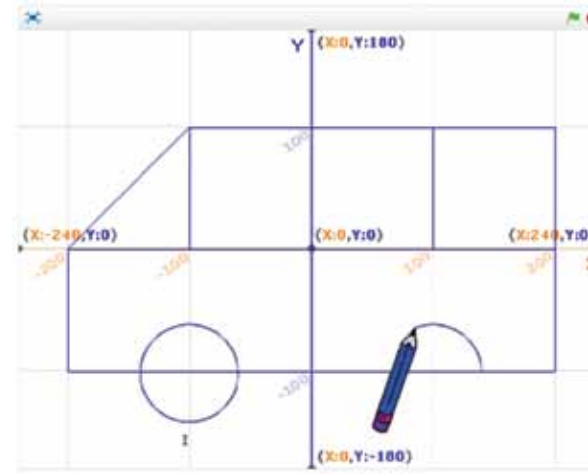


Figure 2: Coordinate geometry and the van.

These projects add a very important element of creativity and beauty and inspire the group as a whole to want to create not just projects, but beautiful projects. It also adds the generally missing element of art to mathematics making learning mathematics all the more joyful.

Getting a sense of variables

"Viewing" an expression, e.g. $(x+y)^2$, as a geometric shape is being encouraged in textbooks in Tamil Nadu as a way of concretizing abstract concepts as squares (x^2 and y^2) and rectangles ($2xy$). These images are usually static in the textbook, but when children create a program for demonstrating this and vary the values of x and y there is a good chance that they will get a sense of the variables and connect it to what they use in programming.

Connect with the real world

Programming hardware allows children to connect with the real world and motivates them to complete a project as they develop problem solving, logical thinking skills and bring these aspects to something they care about. Over the years, the children have made different projects with Arduino (an open source prototyping platform used for building electronic projects) including a school bell, a clock, a display of C3STREAM Land, a water overflow alarm for ground water, a robot that moves things from place to place, etc.

I would like to tell you about a project where a child was having fun creating a two digit clock (measuring seconds) with the seven segment display (a form of electronic display device for displaying decimal numerals).

A couple of the letters in the C3STREAM Land display made by the children a year ago were not blinking properly. Two children in the 7th grade expressed interest in fixing the display. A facilitator walked them through how powering one leg of the display can light up one of the seven segments similar to an LED (Light Emitting Diode). They quickly grasped the idea and fixed the display in a couple of classes. After fixing the display one of the two students was interested in doing something more with Arduino. He wired up the seven segments to different Arduino pins to control the segments individually and managed to write his first program in 'C' to display the number '1' by the end of the class. Using his understanding of blocks in Scratch he extrapolated how he could clear the numbers and change them. He needed some help getting around syntax errors during compiling 'C' programs, but then worked on his own for 45 minutes. In this time, he had made a single digit counter that increased from 1 to 9 every second. He then asked for ideas on extending the functionality to make a clock using Arduino. After talking to a facilitator, he added a second seven segment display to get the second digit of the clock. He even managed to figure out the logic of the first digit continuing to run when the second was implemented to create a 99s counter on his own.

Programming can be an active way to challenge middle school children to find ways to demonstrate their knowledge in mathematics and learn something deeper as well along the way.

TP

Let's explore the expo idea!

A. Ramachandran

The author has taught science, maths and geography to middle school students for three decades. He has contributed several articles on math enrichment to *At Right Angles*, a publication of the Azim Premji University. He may be contacted at <archandran.53@gmail.com>.

What are the shortcomings of our present methods of education (with particular reference to maths)?

i) The teacher has to handle children of varying abilities and temperaments.

ii) The course is driven by the prescribed curriculum, leaving no time for exploration.

iii) Classroom structure is rigid. Students sit in a group facing the teacher, who does most of the talking; paper, pencil, blackboard and chalk – these are the mainstay of the classroom discourse.

More could be said on these lines and it may be difficult to bring about a complete overhaul of the system. But as teachers we should be able to do our bit to tell our students how maths has developed organically down the ages in response to human needs.

One program that I have carried out with the help of other faculty members in a semi-conventional school is the 'Math Expo.' This program works best for classes 6-8. We took a week off from the regular maths curriculum. (Other subject classes were not affected.) The class was divided into groups of 4 or 5, with children of mixed abilities. The physical space was rearranged – tables grouped together so that each group could sit in a circle. Leading questions (see examples below) were given to each group to spur them to discuss, plan, investigate, etc. (The topics had been arrived at after some brainstorming in a prior meeting of the maths faculty.) Students could move freely in and out of the classroom to procure materials for experimentation/demonstration/display. From their observations, students made conjectures which they tested out. The teacher was available to supply them with appropriate terminology, help them reason out things and derive conclusions.

Photo courtesy: John Henry Donovan from FreeImages

Many new facets of the students became evident during this exercise. Students who tended to be quiet in regular classes were now seen to take the lead. Those who had done well by mechanically learning in the regular classes now seemed to take a back seat. There was a lot of self-correction going on in each group. There was more scope for visual and finger skills than in a regular maths class. Each student contributed to the group effort according to his/her capacity. There was no judgmental or comparative angle.

At the end of the week an open house was held inviting all students and teachers of the middle school. The various groups explained their tasks and findings to the visitors with the help of charts and other display materials.

It was found that this experience prepared the ground to introduce new concepts in later classes. Students got a feel for how maths is done – it is not something just 'given' to them that they are expected to follow unquestioningly.

A sample of the topics taken up.

i) **From buckets to parabolas:** The task was to fill a bucket incrementally using a glass/mug, noting the height to which the water rises progressively and making a graph plotting the height against the number of mugs of water added. Students were expected to describe and explain the shape of the graph. Concepts of volume, cross-sectional area,

Photo courtesy: P Widling from FreeImages

etc., got reinforced. In what other situations can we expect to obtain graphs of similar shapes? The task can be repeated with buckets of different sizes and containers of different shapes (barrels, *kudams, drums*…).

ii) **Shadow play:** This involved outdoor work on a sunny day. The task was to measure the lengths of shadows cast by objects such as chair, table, gatepost, goalpost, tree, etc., at the same time of the day. The relation between the length of the shadow and height of each object was noted. The task was repeated at another time of the day. Ideas of proportion and similar triangles are reinforced by this activity. Now a question was raised – can we estimate the height of a tall object from the length of its shadow? A few structures can be looked at in this connection – a tall tree, flag mast, a school block, etc.

iii) **2D symmetry:** Cut-outs of 2D shapes were made (equilateral and isosceles triangles, square, rectangle, rhombus, parallelogram, trapezium, isosceles trapezium, kite, regular pentagon and hexagon, circle, etc.). Their symmetry aspects were noted – reflection axis, rotation axis perpendicular to the plane of the figure (of order 2 to 6), inversion symmetry. Students can look for designs around them and study their symmetry (swastika, rangolis, playing card designs, brand icons, emojis, upper case letters of the English alphabet, etc.).

There is no dearth of topics that lend themselves to such activities – polyominoes, tessellations, patterns in triangular and square numbers, Pascal's triangle, golden ratio, magic squares, Vedic math procedures, platonic solids, maths in sports, etc.

The chief purpose of a math expo is to show that maths is not confined to the classroom but is all around us, if only we have the eyes to see it.

TP

Can we use our participation in PISA to improve our maths learning levels?

Sridhar Rajagopalan and Nishchal Shukla

Sridhar Rajagopalan is co-founder and Chief Learning Officer of Educational Initiatives Pvt. Ltd. He can be reached at <sridhar@ei-india.com>.

Nishchal Shukla is Associate Vice President, Pedagogical Research, at Educational Initiatives Pvt. Ltd. He can be reached at <nishchal@ei-india.com>.

India has agreed to participate in PISA in 2021. A small number of randomly selected students from Chandigarh, the Kendriya Vidyalayas and Navodaya Vidyalayas will write the test next year.

India has participated once in PISA before, in 2009-10, when students from Tamil Nadu and Himachal Pradesh wrote the test. We performed very poorly, ranking 73rd among 74 countries, finishing ahead of only Kazakhstan.

This result was shocking and many people assume that it must have been an aberration. Maybe the students were not prepared for the test or had to take it in English. (Actually, all students were tested in their medium of instruction.) Maybe only government schools were tested – our private schools would have done much better. (A detailed study conducted by our organization in 2006 and repeated in 2012 established that students of our top private schools perform below international average in class 4.)

Our organization regularly conducts assessment for private schools (called ASSET) as well as for various governments. These assessments (as well as tests like Pratham's ASER) seem to suggest that we do indeed have a learning crisis in our system. Our learnings and conclusions based on doing such assessments for about 20 years in India are:

- From a learning perspective, India's education system is very poor – in our analysis falling behind even some of the poorest performers in Africa.
- For India to improve its education system, it needs to a) ensure strong foundational language and arithmetic skills by class 5 and b) have a school leaving exam (board exam) focussed on learning and not simply recall.

What is PISA and can it be more than just a harbinger of (bad) news? Can we learn from assessments like PISA (and good assessments we already have in India) to improve our assessment systems, including our board exams at which all teaching is targeted? In this article, we shall look at what PISA is, delve into its mathematics testing framework in some detail and examine how we can use it to actually improve our learning levels. Since the main problem with our board and school exams today in the poor quality of their questions, we shall take examples from PISA and some other questions and explain why good questions are so critical to attain good learning.

PISA, the Programme for International Student Assessment, is an international benchmarking test that assesses random samples of 15-year old students (typically in class 10) on their reading (language), mathematical and scientific literacy skills. It has been conducted every three years since 2000. A country (or region) must voluntarily agree to write the PISA test. Another international assessment, TIMSS (Trends in International Mathematics and Science Studies) is similarly conducted once every four years for students of classes 4 and 8 since 1995. India has participated only once (2009-10 in PISA) among all these and a few other assessments.

Assessments like PISA and TIMSS have shown that some countries like Singapore, Finland and South Korea have been among the top performers though ranks vary a bit between the years and the assessments. China has been a late entrant but has performed extremely well with Shanghai city topping the PISA in recent years. The US is an example of a country that has spent a lot of money but has performed only at average levels in benchmarching tests since the 1990s.

PISA mathematics – skill framework and types of questions

The PISA tests "are designed to gauge how well students master key subjects in order to be prepared for real-life situations in the adult world."

The test contains items from three core domains – reading, mathematics and science. In each round, one of these is the major domain. **PISA 2021 will have mathematics as the major domain.** This means that there will be more assessment items in the papers assessing mathematical literacy.

The PISA Draft Framework defines mathematical literacy as:
"An individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st century citizens."

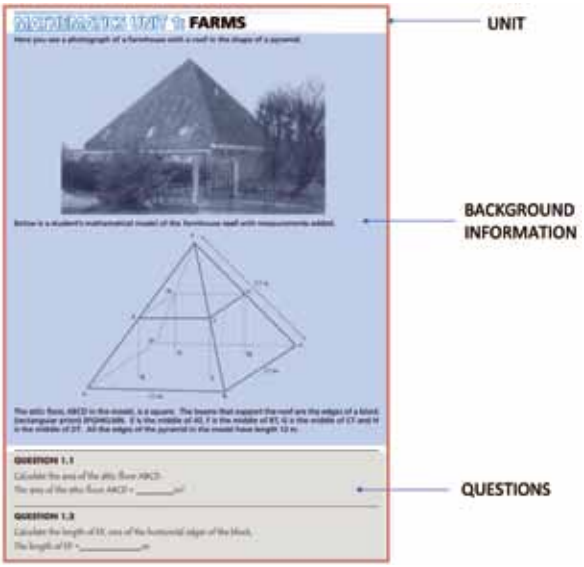
It assesses the following cognitive and content areas.

Mathematical literacy

Cognitive areas	Mathematical Reasoning	
	Mathematical Problem Solving	Formulating situations mathematically
		Employing mathematical concepts, facts, procedures and reasoning
		Interpreting, applying and evaluating mathematical outcomes
Content areas	Quantity	
	Space and Shape	
	Chance and Relationships	
	Uncertainty and Data	

So what differentiates a PISA question from other questions and why do students struggle on PISA questions?

PISA questions require students to a) understand the information provided, b) find what needs to be determined or solved and then c) apply the appropriate procedure. The questions we use in our Indian tests (including board and school exams) tend to be based on a small number of familiar question types. Thus only c) in the above list is tested. That was probably okay for the well-defined world of the 1950s and 60s, but in today's circumstances, the ability to absorb new information and define the problem and choose what needs to be solved, is even more important. Let us look at a few PISA and our own PISA-like questions from ASSET to understand this:



Sample PISA Questions – Mathematics

The purpose of the background information in such questions is to provide a real-life context to the student based on which the questions are asked. That information can be of different types and cover a variety of contexts.

Different types of background information possible:	***Variety of contexts*** background information may cover:
• passages	• personal (individual, peer group, family)
• tabular information/data	• occupational (the world of work)
• images	• societal (local, national or global community)
• graphs or charts	• scientific (natural or technological world)
• combinations of the above	
• some more advanced types of materials involve interactives that PISA uses for computer-based tests	

Importance of good questions and challenges students face

While one may point out that the context used may be unfamiliar and may cause students to struggle to answer the question, it is important to note that this is one of the characteristics of a good question. Many other international assessments including TIMSS and our own ASSET test use questions that are based on unfamiliar contexts or which appear as if they are not something students are used to answering.

For example, consider the following question from ASSET.

Hiten typed in a paragraph about octopuses for a project. His sister Nandita played a prank and changed all the numbers in the paragraph by multiplying each correct number by the same number, say 'n'.

Octopus: a carnivorous marine mollusk, found worldwide in tropical and warm temperate waters. The octopus is characterized by a soft body and **24 arms** bearing **6 rows of suckers** each. As in the vertebrates, the **6 large, complex eyes** of the octopus are camera-like in structure, and their vision is acute. The animals can change the color and texture of their skin rapidly. Many species, such as the common octopus, can grow to about **300 cm long**. A small species, such as the Atlantic Pygmy octopus, however, is only about **6 cm long**.

Now Hiten has to correct the numbers again. What should he put as the actual length of the Atlantic Pygmy octopus? (Hints are given in the visual of the octopus.)

A) 2 cm B) 3 cm

C) 6 cm D) 18 cm

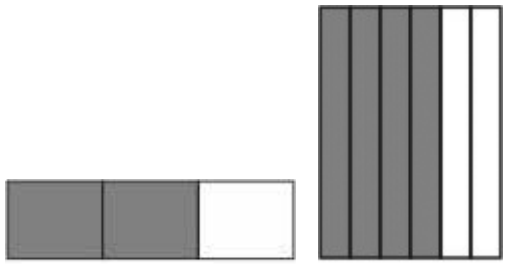
Source: ASSET – https://www.ei-india.com/asset

The question uses a factual piece of information and gives the context of what happened when someone changed the numbers in the factual piece in a certain way. Students are expected to understand this context and use one piece of knowledge that they are familiar with regarding the correct number of arms an octopus has, to solve the problem and answer the question. In fact, even if they don't know that fact, they can take hint from the image given. While a lot of students may find answering this kind of question interesting and the process enjoyable, the first reaction of many teachers is that this is 'out of syllabus' and so students cannot be expected to answer it. In our sample of around 14000 private English medium school students, only 21.5% students were able to answer this question correctly. 31.1% students selected option D, indicating that they didn't really understand the context and the question and blindly selected the length (incorrect) mentioned in the passage.

It is important to note that the data for the two questions above are from students of private English medium schools whose performance tends to be higher than those of government school students. A large percentage of government school students often struggle in the first step itself, in reading the given text.

This argument that students fail to answer due to lack of familiar context fails when we look at their performance on some of the basic conceptual questions like the one that follows. This question was given to the same 14000 private English medium school students. Only 30.7% students were able to answer this question correctly. 27.7% students selected D indicating that they were just used to calculating area of such shapes given their dimensions and could not think beyond that procedural knowledge.

Two sheets of different sizes were shaded as shown.



Which of the following is equal for the two sheets?

1. Area of the shaded part.
2. Percentage of the sheet shaded.
3. Ratio of the shaded area to the unshaded area.

A. only 2
B. both 1 and 2
C. both 2 and 3
D. Can't say unless the dimensions of the sheets are known.

Source: ASSET – https://www.ei-india.com/asset

As seen in the above two examples, a good question is one that challenges and stimulates a child to think deeply and to apply concepts learnt. A good question, correctly framed, can help a teacher understand the thought processes of students and how well a child has internalized a concept or mastered a skill.

Interestingly, a majority of the countries do the PISA test on the computer. The computer-based version of the test includes MCQs (Multiple Choice Questions) as well as other technology-enhanced item (TEI) types. TEIs typically help elicit responses that otherwise are difficult to do in a traditional MCQ or even a free response question format. Here is an example of one such question that allows students to interact with the map, select/deselect a route and check the time taken to travel through the selected route. It then asks a question that uses this interactivity. The advantage of such a question would be that it would capture not just the final answer (which an MCQ could also have) but also what the student selected, the sequence in which selected and the time taken before the student answered the question.

Source: OECD website – https://www.oecd.org/pisa/test-2012/testquestions/question2/

Due to the advantages listed above, the use of TEIs in assessments is increasing. Even ASSET's computer-based version tests students on MCQs as well as TEIs.

Conclusion

Tests like PISA help at multiple levels. While the benchmarking data they provide may serve as a wake-up call, probably even more important is the learning from the type of questions these tests use. If used correctly, a good question can prove to be a highly effective tool in the process of learning and can even enable learning. In that sense, studying tests like PISA and our own tests like ASSET is something every teacher must do. While developing questions like these improve the capacity of teachers, analyzing the student performance shows how children are thinking and the errors they are making. Understanding these and modifying our teaching methods accordingly may be the best way to move towards improved student learning.

TP

What's it about math?

S Sundaram

The author has worked as a principal, teacher trainer and educational consultant in several schools in different parts of India. He retired as the principal of Reliance School in Jamnagar in 2013 and has settled down in Chennai. His areas of interest are primary mathematics, school leadership, quality in education and technology in education. He is currently working on a book on understanding the various concepts underlying all the topics in the K-8 math curriculum. He can be reached at <sundaram021148@gmail.com>.

In the last two decades, three phrases have entered the vocabulary of many math educators – math phobia, math anxiety and dyscalculia. Why is phobia only associated with math? Why don't we hear of history or language phobia? We will try and understand this phenomenon in this article.

There are two inter-related reasons. First is the failure of many educators and teachers to realize that the purpose and content of math are different from that of other school subjects. The second is the inappropriate pedagogy used in math education as a consequence of this.

Math is different

However, most learners intuitively realize that math is different compared to other subjects. But they find it difficult to pinpoint the nature of their difficulties with math. Most can easily describe what they are learning in language, science or social studies, but have difficulty in describing what it is that they are learning in math. Numbers, computations and geometry is what most can come up with.

To understand how math is different, we need to go into the components of any subject that is to be learnt.

What are subjects made of?

The learning of any subject can be classified into the following components – concepts, skills and information.

Concepts are mental abstractions, which need to be constructed in our mind by experience and introspection. Ideas like prime numbers and the operation of subtraction are two examples. They may be mirrored by physical entities and events but cannot be directly "perceived" or "pointed out". You cannot show what "prime-ness" is. We can only point out a few numbers which have this quality. Concepts are the most difficult to learn.

Skills, whether physical or mental, are to be learnt through practice. Performing a long division or colouring a map without crossing boundaries are examples.

Information is "bytes" of information to be remembered or memorized. The names of types of triangle or the birth year of Akbar are examples. We may not even know if a piece of information is true or false. Different pieces of information that we learn, may not even be related to one another. Information is the easiest to learn, as it means "remember" or "memorize".

Each of these components also needs a different way of learning. Information can be delivered through lectures and remembered. Skills, however, have to be practiced and mastered. Concepts have to be understood.

Difference between subjects

Let us represent, in a table, the relative proportion of concepts, skills and information in four common subjects taught in school. The numbers are just indicative. The purpose of the table is mainly to understand the difference between the subjects.

Subject	Language (%)	Math (%)	Science (%)	Social Studies (%)
Information	40	10	50	70
Skills	50	40	20	20
Concepts	10	50	30	10
Total	100	100	100	100

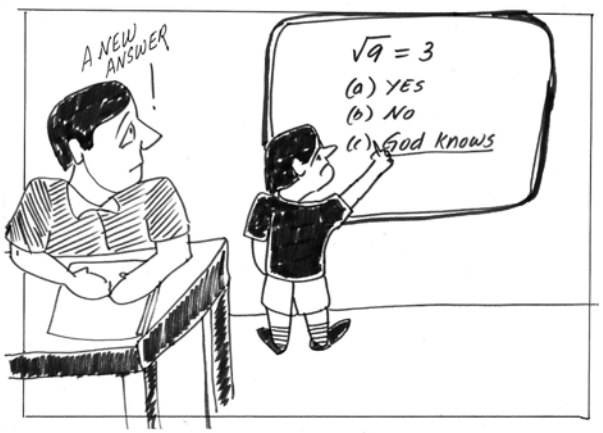
Information is maximum in social studies followed by science. By science we mean science as taught in schools where the emphasis is on information about definitions, discoveries and inventions. So both become subjects with a lot of memorization. Information is the easiest component of learning.

Skills are maximum in language where without listening, speaking, reading and writing skills, a language cannot be mastered. Though skill of experimentation is a major part of science, in schools, laboratory work is not given real importance. Math also needs a lot of skills to be practiced. "Drilling" is a common word used about math education.

Concepts are the most difficult aspect of learning as they involve a lot of introspective thinking and abstraction. Language has the least amount of concepts which is basically grammar and phonic rules. Social studies can include deep concepts, but these do not form part of the school curriculum. Math has the highest level of concepts.

Math is different and difficult

Math is concept heavy. Concepts are abstract mental ideas whose understanding requires a lot of experience, modelling and introspection. Concepts have to be "caught" by the learner. They cannot be directly "taught". These elements make it a difficult subject to understand, and there is no clear understanding of what "understanding" is.



Another difficulty is that all math concepts are related across the curriculum and also related hierarchically across grade levels. We can think of the math curriculum as a pyramid built with playing cards, stacked one on top of another. The weakness of even one card at the lower level will weaken the entire structure. It is similar to the saying that a chain is as strong as its weakest link. A weakness in subtraction will affect understanding of division, fractions, algebra, etc.

How math needs to be taught

Basic ideas in math like numbers, shapes and operations were developed by observing patterns in the world around us. Hence in pre-school and primary school, these ideas can be taught to children using the environment around us and our daily life experiences.

But very soon, these basic concepts start developing into complex ones. Continuous layers of abstraction get added on like the layers of skin on an onion. Hence along with teaching of these basic ideas, the very process of thinking about these ideas and the inter-relationships between them should also be developed. If these skills are not developed early on, these concepts will become so abstract that students will not be able to understand them.

Beginnings of math anxiety and phobia

If the teaching of math is not appropriate, the students' lack of understanding keeps deepening, as the complexity of the concepts keeps increasing. A learned psychologist has compared the mental state of a student who cannot understand what is happening in the class to that of a novice swimmer who struggles just to keep his nose above water!

In addition, negative comments about weaker students reduce the self-confidence of students. Hence, each math class brings a lot of anxiety. Anxiety itself reduces the motivation to learn and hence further reduces the capacity to focus and learn. For many students the level of anxiety keeps building up leading to the psychological condition of a phobia.

What is the way out?

Since math phobia is a condition which develops over a number of years, there is no short-term solution

to it. In the long run, there are several interconnected strategies which could be adopted.

Appropriate pedagogy

Concepts cannot be directly pointed out. They have to be demonstrated in a variety of indirect ways so that the student can catch them. My guru used to say "Concepts are not taught. They are caught". So the concept of an even number should be shown to students by equal sharing which is a daily event they are familiar with. This can be done using simple manipulatives. Just providing a definition (that an even number is divisible by 2) may not produce understanding. It can even mislead as this statement is not true if numbers are written in odd number bases!

A math activity centre with plenty of manipulatives and models is extremely necessary for children to "practice and explore" math.

Some examples of skills in math are computations and constructions in geometry. In general, skills have to be practiced in a variety of ways. But math skills are intimately woven with underlying concepts. They have to be practiced with understanding of the concepts. Mastering addition of two digit numbers needs a clear understanding of the place value concept. Practicing skills mindlessly may lead to mistakes when the procedure needs to be modified to suit the situation. My guru used to say "drilling may produce only holes and not understanding."

Information in maths is mostly the symbols, names and definitions. Many terms used in math are never heard by students outside the math classroom – obtuse, coprime, etc. Some have different meanings when used in daily life – interest, imaginary, etc. In such cases, students have to be helped in absorbing these terms and their connotations by repeatedly using them in context in class for extended periods of time.

Appropriate age-wise pedagogy

Children do an average of 15 years of schooling, during which they go through several stages of mental and emotional development. Developmental psychology tells us that until the age of 10, their ability to understand abstraction is limited.

Hence as children proceed through different class levels, the method of teaching should change to be in consonance with their mental ability to learn complex concepts, skills and information.

Unfortunately, we see that, in our schools, all subjects at all class levels are taught using the same process – lecture, blackboard, chalk, duster, students cramped in rows, textbooks, classwork, homework, examinations, pass/fail, report cards.

Teacher training

Such changes in pedagogy require a sea change in our teacher development and certification courses and on-the-job training. In our schools, continuous teacher training is almost never heard of!

Primacy of the primary

The strong foundation for developing motivation and competency to learn math has to be laid in primary school. In primary school, all skills are important. Hence in primary school the curriculum has to be reduced and teachers have to ensure that each and every child who leaves primary school attains at least 80% understanding of the entire curriculum. A pass mark of 40 or 50% is inappropriate in math in primary school.

Assessments

Formal examinations, which reduce the number of teaching periods, should be replaced up to primary with class tests and that too of the formative type. Methods of assessment should be broadened to include oral, practical and project assessments. Timed tests should be reduced to the minimum. The textbook should become a reference, rather than the "bible". Mistakes should be seen as "learning opportunities" and not "shaming opportunities"

Math is critical

The world has changed dramatically in the last five decades. In this digital and knowledge society all children must be given an opportunity to learn and enjoy math which is going to be extremely necessary for them to lead an empowered life.

TP

The gentle man who taught infinity

Sheshagiri K.M Rao

The author works as an Education Specialist with UNICEF. Though he began his work as a Mechanical Engineer in 1991, his keen interest in education led him to become a schoolteacher a few years later. Since then, his work in education has spanned a variety of contexts with local, national and international organizations as well as with the government. Over the years, he has worked with children, developed educational interventions in diverse contexts, facilitated processes of organizational reflection, has done educational research and written on contemporary educational issues. He is currently working on his second book, which is a work of history located in education. He can be reached at <giri.shesh@gmail.com>.

"What does the protagonist of *The gentle man who taught infinity* have to say about the teaching-learning of mathematics?" Before I answer that question, a little bit of the story…

It would be apt to begin with a quote by American historian, Henry Adams: "A teacher affects eternity; he can never tell where his influence stops." My relationship with Mattur Venkatadri Channakeshava, the bespectacled gentleman who taught me math in school, is a lot like that.

Thirty years after I graduated from the Baldwin Boys' High school, Bengaluru, I decided to write about him. I felt this deep urge to reconnect with him. I wonder why it took so long. After all, I must have passed by his house many times in all these years. I could have met him as many times as I wanted to, but I didn't. I guess it is all about waiting for the right day.

As Adams says, we pick up many things from our teachers in school, things which affect us in ways that we can only understand much later. And what we pick up keeps visiting us time and again, in ways we cannot foresee.

I wrote about Channa because of two impulses – hope and anger. I believe they are a potent combination. I'm angry because our schools are letting our children down. Survey after survey by the government of the day or those outside the government, shows the abysmal depths to which we have managed to plunge. We cannot sink any further. The only way is up. The crisis of learning is hidden, eating away at our vitals as a society. Imagine – a child who reaches the 8th grade carries many years of 'learning deficits' in all the subjects – what she manages to learn, in effect, is only the equivalent of the 3rd or 4th grade! What a waste of her time! I'm not getting into a longwinded discussion on why this is happening. The undeniable fact is that it is staring at us, challenging us to respond.

I'm hopeful because I have started telling Channa's story. There is much that we can imbibe from it to make math learning enjoyable. It is now a drudgery for many, with the robotic and mechanical ways of teaching that we have perfected over a long time. Stories like Channa's, I hope, will help us come out of those miserable depths. And out of 'math-phobia,' which afflicts many. At least then, children might look forward to school.

I have chosen to write about an extraordinary teacher, his subject and his craft. I want to

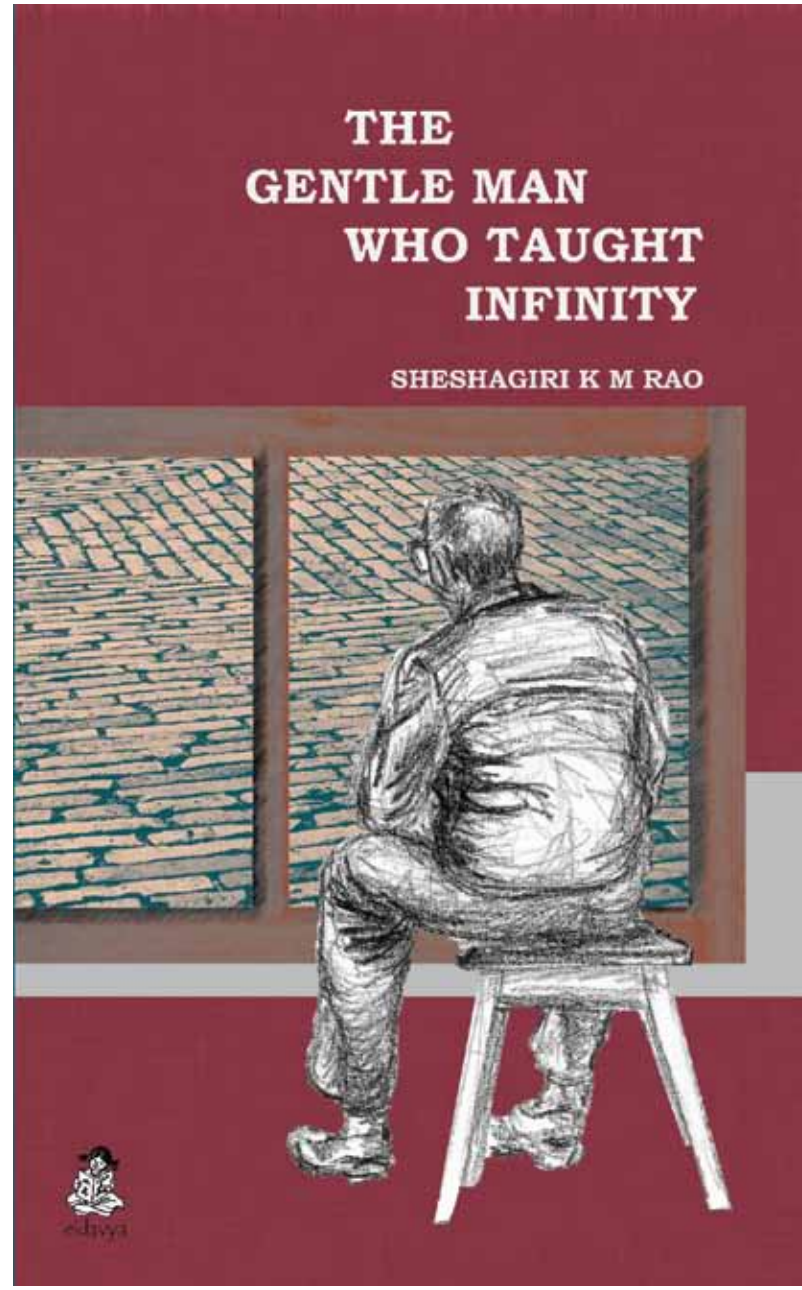
spread the word, so that we all – teachers, parents and those interested in education – can learn a thing or two from him. Stuff that we can do and enjoy. And see the difference. I believe this would be a useful and positive thing to do.

I was fortunate to have learnt high school mathematics from Channa. He didn't make me a rank holder or gold medalist in the subject. No, he didn't. On the other hand, I was just an average student of the subject and very nearly failed in it in the ninth grade. But he made me love the subject. That is what he wanted to inspire in all of us, even in a school that was conventional and which prided itself on exam results, a school where corporal punishment was rife. So, let me set out the few things he did which I remember to this day.

Channa made us enquire and think. He was always gently nudging, pushing us to figure things out on our own. His teaching was always based on first principles (read ahead to know more). You never got a sense that he was ad hoc. He taught us to see a pattern in the madness. Channa taught us mathematics because he wanted us to pursue truth and beauty, not mere exam grades. Yes, you heard me right! Never mind if none of us became mathematicians. For me, he made math fascinating. Finally, Channa was a great hit because he was a master storyteller who loved his craft. He is the eternal student of the subject who carries within his being a deep understanding of its structure, mysteries and paradoxes. These, he happily shared with us. Which is why, I remember his teaching even after so many years.

So, we stood on his shoulders and looked beyond, enjoying the mathematical landscapes that were made visible. Beautiful and often startling landscapes.

Channa took us on a roller coaster ride of the world of mathematics. Beyond the syllabus and our sterile textbooks, from cyclic numbers to Euclid to Gauss to Bhaskaracharya to Cantor to Euler and the bridges of Konigsberg, to Barber's paradox and Fermat's last theorem and the four-colour problem, interspersed with discussions on infinity, Channa allowed us to glimpse the real mathematics. These intriguing patterns, problems, mysteries and paradoxes which are the heart of mathematics, he said, need to be understood by us as students to truly appreciate and love the subject. The best part was how they were intimately linked to what we were studying in school. The sad part was that



they were not included in our syllabus. Our curriculum and syllabus makers deprived us of the best parts, year after year. In our examination centered system, the joy of learning and discovery is lost. Channa reminded us that this need not be so, even in a regimented school.

Suddenly, mathematics had a very human story, lots of meaning, and a character of its own. Channa demonstrated how even the most complex ideas of mathematics could be made simple for kids like me. I wanted to know more and more. So what if I was a so-called average math guy?

So, what does the protagonist of *The gentle man who taught infinity* have to say about the teaching-learning of mathematics? While reflecting on this question, I would like to first point out that the teacher, his craft and his subject are inseparable. Therefore, the lessons that can be culled out from Channa's story straddle these three elements. I have outlined a few below:

- Engaging with the discipline as a teacher – Channa best exemplifies the aspect of a teacher who loves the subject (this also came from childhood and later influences), being curious about the subject, being a lifelong explorer and student of the subject.
- Using the story as a medium of transaction – Channa generated and kept the interest going by using stories. He had a deeply intuitive understanding and appreciation of 'pedagogical content knowledge'.
- Using history along with the story – Channa used history as one of the key underpinning elements of his pedagogy. He was as good a history teacher.
- Through the use of story and history, Channa helped us 'locate' the topic. By doing this, he provided the perspective that math has not descended from hell, that there is a human history to it, that there are connected ideas; that there is a process of development of these ideas (messy though, the process may be, full of human elements); that many things taught in the classroom have their own histories of development.
- Teaching based on first principles – for Channa, the purpose of using history and the medium of the story was to develop understanding and perspective. Teaching using first principles meant that he was looking at developing understanding and insight, which are critical requirements for any learning. Channa nudged, prodded, promoted struggle; he would not just 'give away' the answer and he would often wait to see what happened as we were learning. Some of the best examples of this 'teaching based on first principles' came from our learning of geometry.
- Everyone should come on this journey – no excuses here! The issue of equity was quite central to his scheme of things.
- Valuing the child's ability – the *belief* that every child can understand and appreciate deeper stuff – that is why he went beyond.
- Being gentle and patient... he completely eschewed corporal punishment in a school where it was rife.
- 'Using the blackboard space beautifully. Channa's writing on the blackboard was always a treat to behold – beautiful handwriting and a very aesthetic sense of proportion. He used the entire blackboard space to powerfully illustrate his points.

Finally, I would like to say that there is much that we can learn from Channa to enrich the teaching-learning of mathematics at school, especially during times where acing the examinations seems to be the preoccupation of schools, teachers, parents and students. When you do lay your hands on the book (*The gentle man who taught infinity*), jump into the fray. What else could be more important?

TP

Why Maths?

Aditi Mathur and Ratnesh Mathur

The authors run an open unschool called Aarohi and invite all readers to visit and see how open learning can be an amazing way to work with children. They also conduct training retreats and online training for teachers and parents. Visit http://www.aarohilife.org to know more.

What if maths was optional – an extracurricular subject – much like music? What would happen if the primary purpose of being exposed to maths, akin to music, was to enjoy and appreciate it (and not necessarily learn it)? Most of us do not learn music and yet most of us enjoy some form of it.

Some children like maths naturally, some don't. Just like some like sports naturally, some don't, some like reading naturally and some don't and some like loads of friends while some like to be by themselves.

When it comes to music, most of us sing out of tune/rhythm; we do not mind it and we still sing (in bathrooms and elsewhere!). If people don't like our singing, at the most, they may ask us to sing in private.

What would happen if similarly we were allowed to do maths out-of-tune? And if by chance people did not like our math-mistakes, they just let us commit them in private.

When we need accuracy, beauty and quality in music, we switch on our MP3 players and listen to songs by professionals/musicians. What would happen if similarly we were allowed to – in maths – when requiring accuracy and quality – resort to calculators, apps or even take the help of professionals/mathematicians?

Photos courtesy: Ratnesh Mathur

Most of us do not learn music (as in go to classes) because that is not how we use it in our lives. You see, in our lives we listen to music. What would happen if similarly – in maths we were to know only as much as we need in life? Now most of us – in our whole life – use the maths we learned in 3rd or 4th grade – the four operators, measurements and to some extent percentages. Which means if at all we want to teach maths, we can make it optional after 4th standard!

Also, like music, we can learn numbers while playing games or shopping (most of us like either of these activities and many a times both). Most people in the village where our campus is located have not gone to school but have a sufficient knowledge of numbers to live their lives comfortably.

Sometimes people say we need to learn maths to develop critical and logical thinking. Is maths the only way to develop critical and logical thinking? Why not games like chess or Battleship or Monopoly? And do we conclude that our humble elders living in villages lack critical and logical thinking? In fact, they can observe phenomena in their fields and homes and think about them in terms of cause and effect and predict and estimate sometimes better than those of us who are educated!

Maths in itself is beautiful – and like any other art gives joy to its explorer – but then, shouldn't we (and children) be allowed to choose which art we want to explore? Some like to trek mountains, some like to trek algorithms – both are incomparable and beautiful and useful and joyous in their own ways.

Just as we need not make everybody like art or sports or politics or history, we need not make everybody like maths. We can let children and people live out of their choices!

What do you say?

Why not music?

What if music could replace math?

Obviously the assumption is we would not like to add another subject as mandatory and hence replacement is the only way. Also I agree that basic maths – up to 3rd/4th standard is indeed required to equip everybody with basic computation (day-to-day math) skills.

So let's talk about music

Almost everybody likes to listen to music. If something is so universal and considered by most as divine, we need to seriously consider its role in the development of children.

Considering music's sensorial, emotional, social and mental prowess, how about a daily dose of music? Imagine a child is exposed to 30 minutes (a period) of music every day from the 1st standard to the 10th – this could include singing together (anything from classical *aalaps* to rap songs) or drumming (drum circles are super popular among children), playing instruments or even dancing together to some music.

Three aspects stand out:

1. Music can be experienced together – whole group – a completely different chemistry and synergy and resonance.
2. An individual need not be accurate or harmonious or perfect – everything goes – joy is the real purpose.
3. It is kinesthetic, visual (loads of imagery) and auditory. It incites verbal, mathematical, social, self, body and yes musical – six out of eight multiple intelligences. It's a complete mind-heart-body-soul workout.

With music, there need not be any examination, grading, or comparison. Those who want to take it forward can get into advanced training, while the others can continue their half hour dosage of energy and bliss.

Let's give children choice, while their inner beings soar.

TP

Hanuman's tail – continued fractions and Ramanujan

V S S Sastry

The author is a math communicator. He has experimented with all types of communication tools available to take maths to people. He creates math cartoons, builds math models, does glove puppetry for math tales, teaches math activities through origami, designs popup books, and writes scripts for math videos. He can be reached at <vsssastry@gmail.com>.

P. C. Mahalanobis, who was a student at Cambridge and who later became an eminent statistician and founder of the Indian Statistical Institute, Kolkata, was a friend of Srinivasa Ramanujan. One day, while visiting Ramanujan, he read about a problem in Strand Magazine. Ramanujan was in the kitchen, cooking. Mahalanobis tried to solve the problem and thought he would ask his good friend about it. So he turned to Ramanujan and said, "Here's a problem for you." "What problem? Tell me," said Ramanujan, still stirring the vegetables. Mahalanobis read out the problem.

A certain street has between 50 and 500 houses in a row, numbered 1, 2, 3, 4, ... consecutively. There is a certain house on the street such that the sum of all the house numbers to the left side of it is equal to the sum of all the house numbers to its right. Find the number of this house.

Ramanujan: Did you find the house number?
Mahalanobis: Yes, I did. If the street had 15 houses, then the required house number is 6, since the sum of all the house numbers to the left side of it is 1 + 2 + 3 + 4 + 5 = 15 which is equal to the sum of all the house numbers to its right 7 + 8. But unfortunately, this doesn't provide the required answer since it is given that the street has between 50 and 500 houses. I got stuck here.

Ramanujan: That means there are many solutions to the stated problem isn't it?
Mahalanobis: Yes, I think so.

Ramanujan immediately said: Take down my solution which is in the form of continued fraction.
Mahalanobis: What? A continued fraction?

Ramanujan: Yes, take it down.

$$6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6}\ldots}}}}\ldots$$

Ramanujan: This continued fraction provides all possible solutions (infinitely many) to your problem if you ignore the condition of 50 to 500 houses.
Mahalanobis: Oh, I see, let me see what is going on?

$\frac{6}{1}$. First convergent fraction giving the initial

solution that Mahalanobis found.

$6-\frac{1}{6}=\frac{36-1}{6}=\frac{35}{6}$ Second convergent fraction.

$6-\frac{1}{6-\frac{1}{6}}=6-\frac{6}{35}=\frac{210-6}{35}=\frac{204}{35}$

Third convergent fraction.

$6-\frac{1}{6-\frac{1}{6-\frac{1}{6}}}=6-\frac{35}{204}=\frac{1224-35}{204}=\frac{1189}{204}$

Fourth convergent fraction.

Here 6, 35, 204, 1189, . . . (the numerators of successive convergent fractions) are the required house numbers. Thus after 6, the second possible house number must be 35 since 1 + 2 + 3 + . . . + 34 = 595 = 36 + 37 + . . . + 49. But again this doesn't fit the solution asked for in the Strand magazine problem, since here, there are only 49 houses in the street. Hence we need to consider the third convergent fraction.

The third convergent fraction having the numerator 204 must be the correct solution to the Strand magazine problem since 1 + 2 + 3 + . . . + 203 = 20706 = 205 + 206 + . . . + 288. This solution suggests that the street has a total of 288 houses (satisfying the condition between 50 and 500 houses) in which the desired house number must be 204.

Note that if we do not restrict ourselves to the condition that the street has between 50 and 500 houses, then we get an infinite number of solutions, thanks to the continued fraction provided by Ramanujan.

That was Ramanujan. Quick in answering. And he used

continued fractions with ease. Look at the dots in the above solution. It means fractions continue to be written endlessly. Hence the name continued fraction. If an analogy can be given, it continues like the tail of Hanuman. Ramanujan was fascinated by this Hamuman's tail – continued fractions. His notebooks reveal that nearly 20% of his jottings are about this.

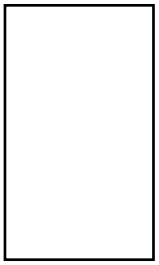
Why was he so attracted to this type of writing? We do not know. But it looks beautiful. It is compact (as in the example above), but gives out infinite solutions.

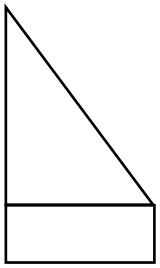
But this beauty, compactness and infinite answers in its womb (continued fractions) is denied to our students. Because continued fractions as a form are dropped from the textbooks. Our teachers and students know or are exposed to only the math of the textbook. That's why they cannot appreciate this episode that illustrates Ramanujan's brilliance. What a tragedy!

We need not go in search of problems for continued fractions. They are everywhere. Even in an ordinary A4 size paper.

Again the story goes like this

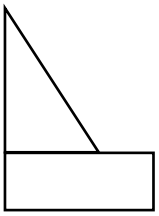
Take an A4 size paper P-1

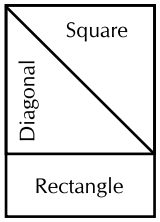




Fold a right angle triangle P-2

Fold the remaining paper up P-3





Open the paper and it looks like this P-4

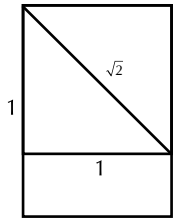
There is a diagonal inside a square. A square has equal sides. We can consider the side to have value 1. Then diagonal = $\sqrt{1^2+1^2}=\sqrt{2}$

(Pythagoras theorem)

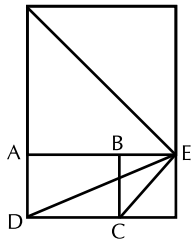
This is exactly the breadth of the A4 size paper. The length is the side of the square. That means the aspect ratio L:B = 1: $\sqrt{2}$. That's why an A4 size paper is also called $\sqrt{2}$ paper. Now-a-days, it is called a silver rectangle (compared with the golden rectangle 1: 1.618).

Now look at our silver rectangle. We have these measures

Remainder rectangle P-5



Inside the remainder rectangle you can fold a square and its diagonal.



Observe ΔCED is an isosceles triangle (CE = CD). That means excluding the square the remaining rectangle (ABCD) is also a silver rectangle. (*Note: This may be verified by taking AE as 1 unit and working out the other relevant sides using the Pythagoras theorem.*)

As earlier, we can fold a square in this rectangle and as earlier a residual rectangle exists. You can continue this process again and again. Every time a residual rectangle remains.

Now let us go back to the A4 rectangle.

Here L x B = 1X $\sqrt{2}$ = square + remaining rectangle
= 1X1 + remainder

$\sqrt{2}$ = 1 + remainder

As silver rectangles repeat, their aspect ratios (L:B) are maintained. They are similar rectangles. Hence the ratio of length (= 1 unit) and breadth (= 2sq + remainder) is maintained for all. Rectangles get smaller and smaller but retain the aspect ratio to the mother rectangle.

So we get an equation

$$\text{Remainder} = \frac{1}{2+\text{remainder}}$$

Substitute in previous formula

$$\sqrt{2} = 1+\frac{1}{2+\text{remainder}}$$

$$= 1+\frac{1}{2+\frac{1}{2+\text{remainder}}}$$

$$= 1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\text{remainder}\ldots}}}$$

This is continued fraction for $\sqrt{2}$. Let us see what we get when we consider convergent fractions at every level

$$1\frac{3}{2} = 1.5, \frac{7}{5} = 1.4, \frac{17}{12} = 1.4166$$

$$\frac{41}{29} = 1.4137, \frac{99}{70} = 1.4142\ldots$$

The last one – $\sqrt{2}$ = 1.4142...is recognizable.

This is the beauty of continued fractions.

TP

The not-so-hidden aesthetics of maths

B S Shylaja

The author specializes in observational astrophysics. After obtaining a PhD on hot binary stars, she continued research on comets, peculiar stars and infrared astronomy before joining the Planetarium at Bengaluru. She has been working in the area of history of astronomy using unconventional records of stone inscriptions. She can be reached at <shylaja@taralaya.org>.

The title perhaps reminds one of the famous golden ratio; the ratio that determines the most pleasing aspect ratio for a post card, a greeting card, posters or a simple visiting card. There are many more ideas from mathematics that have defined aesthetics from ancient times.

When I was a child I remember my seniors posing a puzzle to our class that involved transporting a cow, a tiger and a bag of fodder for the cow across the river. I will change the components so that you will not be bothered by clever questions from your students (How can you transport a tiger? Was the cow not scared? The tiger can swim … and so on, questions that we never thought of asking at the time).

So, here is a farmer with a dog, a hen and a bag of grains. He has to cross the stream with a boat which can carry only two items at a time. Of course, one is the farmer himself because he has to row the boat. Suppose he takes the dog first, the hen will tear the bag to eat the grains. Suppose he takes the bag of grains, the dog will kill the hen. So how will he manage? He is allowed to make any number of trips so that he will leave one item on the other bank and return to pick the others.

I hope you have successfully figured out the answer without much effort. Such exercises can be used to teach "permutations and combinations". It is not just in the classroom but this kind of thinking is being employed in the manufacturing of combination locks, allocation of telephone numbers and vehicle registration numbers and even in the generation of automatic passwords.

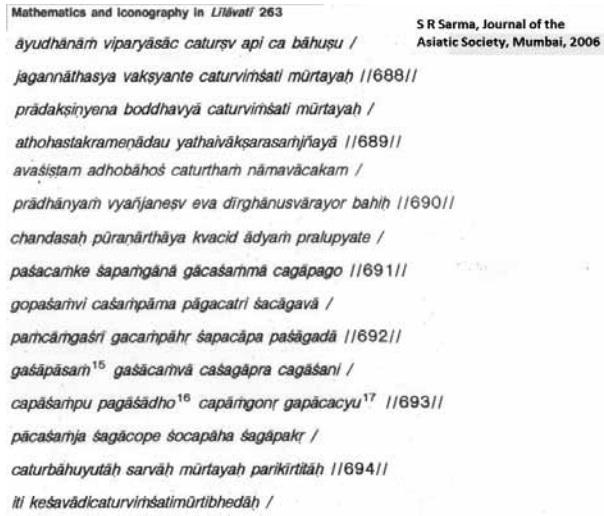
It is interesting to note that it is not just our new textbooks but even the (really) old texts of *Bhāskarāchārya* and *Mahāvirāchārya* discuss derivation as we learn it today and provide practical examples. One example concerns the arrangement of letters in an order called *Candas*, translated as meter.

Let us address a problem faced by sculptors. The sculptors were approached by temple trustees wanting an idol either of *Keśava or Janārdana or Narayana*. All are different names of *Viṣnu*, but the sculptor has been taught that each of these idols has to be made according to the given rules. The idol has four hands, each of which carries a different item – *Śankha, Cakra, Gadā and Padma*. The name specifies the items that should go in each hand. For example, *Keśava* has *Padma* in the lower left hand, *Śankha* in the upper left hand and *Chakra* in the upper right hand (the remaining item in the fourth is understood). Narayana will have *Śankha* in the lower left hand, *Padma* in the upper left hand and *Gadā* in the upper right hand. The list is available in many texts devoted to iconography. Given this how many such idols are possible?

This is a simple problem that can be solved using "permutations and combinations". There are four items (A B C and D) and four places (a b c and d). Staring with (Aa Bb Cc and Dd), (Ab Ba Cc and Dd)

etc., we get 24 possibilities. If we start writing down all the possibilities it will fill multiple pages and each time we want to look up one possible combination we will have to refer to every single page. So how did the sculptors manage to remember this order? That too for centuries? The clue comes from the fact that the same rules are applicable all over India.

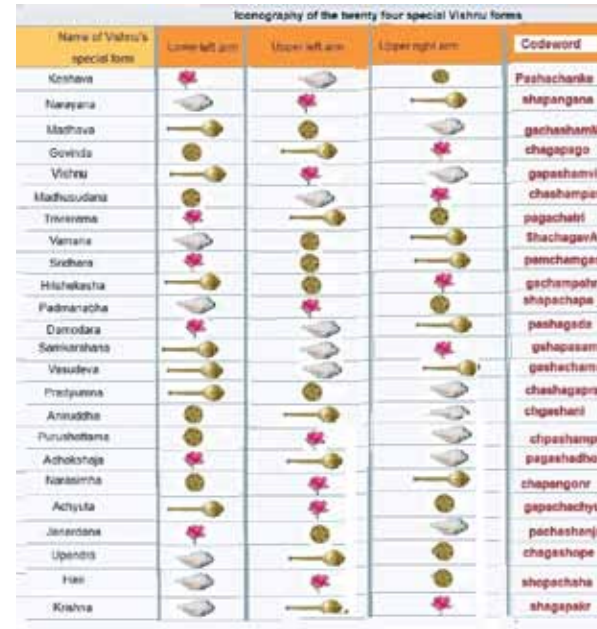
This secret was uncovered by Prof. S R Sarma, a veteran historian of mathematics and astronomy. He brought to light a book by the name *Manasollāsa* written in the 11th century which has a verse set in proper meter (*Chandas*) and can be memorized. It is coded very elegantly giving the first letter of the three objects in the order of lower left, upper left and upper right. The first phrase is *paśachanke* means *pa (Padma) sha (Śankha) cha (Chakra)* and the fourth syllable is for the name *ke (Keśava)*. You can appreciate the cleverness in just specifying three to match the meter since what has been left out is obviously the fourth item. The second example is the phrase *śapangana – sha (Śankha) pa (Padma) ga (gadā) and na (Nārāyana)* and so on. Here is the verse reproduced from the work of S R Sarma and the list.



S R Sarma, Journal of the Asiatic Society, Mumbai, 2006

āyudhānāṁ viparyāsāc caturṣv api ca bāhuṣu /
jagannāthasya vakṣyante caturviṁśati mūrtayaḥ //688//
prādakṣiṇyena boddhavyā caturviṁśati mūrtayaḥ /
athohastakrameṇādau yathaivākṣarasaṁjñayā //689//
avaśiṣṭam adhobāhoś caturthaṁ nāmavācakam /
prādhānyaṁ vyañjaneṣv eva dīrghānusvārayor bahiḥ //690//
chandasaḥ pūraṇārthāya kvacid ādyaṁ pralupyate /
paśacaṁke śapaṁgānā gācaśaṁmā cagāpago //691//
gopaśaṁvi caśaṁpāma pāgacatri śacāgavā /
paṁcāṁgaśri gacaṁpāhr śapacāpa paśāgadā //692//
gaśāpāsaṁ[15] gaśacaṁvā caśagāpra cagāśani /
capāśaṁpu pagāśādho[16] capāṁgonr gapācacyu[17] //693//
pācaśaṁja śagācope śocapāha śagāpakr /
caturbāhuyutāḥ sarvāḥ mūrtayaḥ parikīrtitāḥ //694//
iti keśavādicaturviṁśatimūrtibhedāḥ /

The verse with the codes for the idols



The list of names and associated items in the four hands and the code word

However, one may naturally think that there should have been one standard depiction of all the 24 somewhere as a reference like a textbook. When the sculptors had the time, money and freedom to execute such a task, they did it. A small village called Nuggehalli, which is very close to Beluru, with the temple famous for its sculptures (in Hassan district of Karnataka and a well-connected tourist destination), houses all the 24 statues in the same order as depicted in the verse.

Let us now see an interesting application of the same in Indian music. In all ragas we have seven notes (*svara*). It is well-established that the names of the *rāgās* have been coined so as to represent the serial number. The melody of the *rāga* is governed by the choice and rendering of the *svarās* and musicians excel at this. The *rāgās* are derived from the same seven notes with all possible combinations. Yes, but how many?

Bhāskarāchārya poses a very interesting problem in *Lilāvati*, assuming that everyone knows that there are 24 forms of Vishnu. It reads like this (from S R Sarma)

Just as in the case of *Viṣnu*, can you figure out how many forms of *Śambhu* are possible? He has 10 hands and 10 emblems.

pāśāṅkuśāhiḍamarūkakapālaśūlaiḥ
khaṭvāṅgaśaktiśaracāpayutair bhavanti /
anyonyahastakalitaiḥ kati mūrtibhedāḥ
śambhor harer iva gadārisarojaśaṅkhaiḥ //263//

"How many permutations will there be of the iconic forms (*mūrtibhedāḥ*) of god *Śambhu*, by the exchange of his ten emblems, namely, the *pāśa* (noose), the *aṅkuśa* (elephant's hook), the *ahi* (serpent), the *ḍamarūka* (tabor or small hand drum), the *kapāla* (skull), the *śūla* (trident), the *khaṭvāṅga* (staff with a skull at the top), the *śakti* (spear), the *śara* (arrow) and the *cāpa* (bow) in his different hands, like those of Hari by the exchange of the *gadā* (club), the *cakra* (discus), the *saroja* (lotus) and the *śaṅkha* (conch) ?"

It is mandatory that a *Sampūrṇa rāga* should have *sa (Ṣadja)* and *pa (pancama)*. Then we are left with the other five to be chosen. A definite set of ratios are selected in fixing the other notes. *Ṛṣabha, Gāndhāra, Madhyama, Dhaivata and Niṣāda* are the names given. As shown in the figure, two Ṛṣabhas are possible represented by R and r; two *Gāndhāras* are possible G and g; similarly M and m, D and d, N and n are marked. Let us attempt the combinations:

SRGMDN can be one *rāga*
SrGMDN can be another
SRgMDN
SrgMDN… and so on.

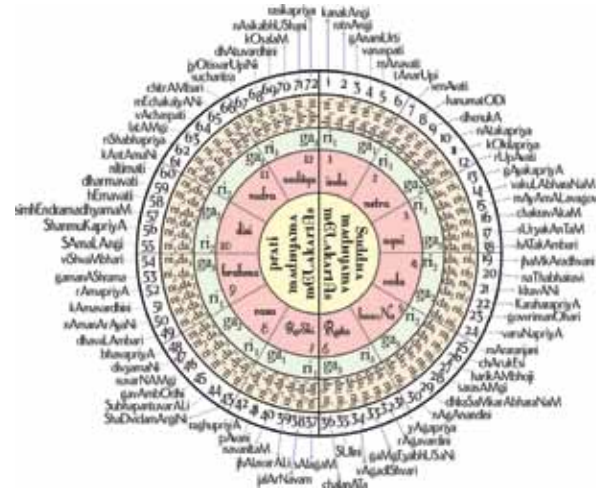
So how many ragas are possible? We need to pick one each from the two possible notes for R, G, M, D and N. That makes it 2 x 2 x 2 x 2 x 2 = 32. Therefore, we expect that there should only be 32 *Sampūrṇa rāgās*.

Yes, that was indeed the case. There are several references to 32 *rāgas* in the Hindustani system. Dr Sukanyā Prabhākar, a well-known musician, has studied the old texts and found that this system of 32 was prevalent till about the 15th or 16th century. We find references to that in the compositions of the great saints *Nijaguna Śivayogi* and *Purandara dāsa*. However, the identification of 32 *rāgas* appears to differ over centuries.

An interesting application of permutations and combinations can be seen in a text of *Jaina* philosophy. There are 15 kinds of *pramādās* defined; *pramādā* can be roughly translated as carelessness or mistake or an offence. There are five subdivisions with unequal distribution; they are *vikathā* (wrong talk, 4), *kāṣāya* (passion, 4), *indriyā* (sense, 5), *nidrā* (sleep, 1) and *praṇayā* (attachment, 1). Combinations of five are made taking one from each category. The *Jaina* philosopher Nemi Chandra of the 10th century has laid down the rules and provided serial numbers also for them. How many such combinations of offences are possible? It is simply 4 x 4 x 5 x 1 x 1 = 80. The allocation of the serial number also is quite intriguing; one may read it from his work *"Gommatasāra"*.

There is another option possible. We can consider three options for R and D. Essentially, R can have R, r and R′. Here R′ is the same as G. Similarly D can have D, d and D′ where D′ is the same as N. So the new *rāgas* can be
SR′gMDN and
SR′gMD′n and other combinations.

This gives us a total of 3 x 2 x 2 x 3 x 2 = 72 *rāgas*. This system of *melakartās* is very well known today. All the combinations result in melodious compositions and are quite popular. The subdivisions into 12 chakras are also standardized.



(Image courtesy: Basvarajtalvar: https://upload.wikimedia.org/wikipedia/commons/0/0e/Melakarta.katapayadi.sankhya.72.png)

Here we discussed two examples of application of permutations and combinations in an apparently unconnected field. If we look at nature which has an amazing variety of biological and plant species, we are sure to find more examples – maybe in the patterns of leaves or sequencing of proteins.

TP

Rewriting math textbooks

Swati Sircar

The author is Assistant Professor at the School of Continuing Education and University Resource Centre, Azim Premji University. Math is the second love of her life (first being drawing). She is a B.Stat-M.Stat from Indian Statistical Institute and has an MS in math from the University of Washington, Seattle. She has been doing mathematics with children and teachers for 12 years and is deeply interested in anything hands-on – origami in particular. She can be reached at <swati.sircar@apu.edu.in>.

In 2018, Sikkim wanted to revamp its primary textbooks for language, mathematics and environmental studies (EVS). The state followed NCERT textbooks from class 6 onwards and was not satisfied with the existing textbooks for classes 1-5. They took help from the Mahatma Gandhi Institute of Education for Peace and Sustainable Development (MGIEP), UNESCO and Azim Premji University for this venture. MGIEP was trying to embed education for sustainable development (ESD) in textbooks. This was the first time such an embedding was being attempted for primary level textbooks anywhere in the world. The state had also partnered with Jodo Gyan in the past. So, most of the resource persons were exposed to how interactive and fun mathematics can be when various teaching-learning aids [mat(h)erials] are used. The team of authors included teacher-educators from the SCERT and DIET(s) and school teachers.

The mathematics groups had 12 authors divided into three equal groups for classes 1-3. Dr. Wagner from UNESCO and two of us from the university worked with these dozen authors. The first few meetings were to orient the authors to various aspects of textbook writing, give them an understanding of ESD and also to develop a syllabus for these three classes. We met for almost a week every month once the chapter writing started. The Sikkim SCERT had already received a set of the number kit (for whole numbers, fractions, decimals and integers) from Math Space, the mathematics laboratory of the university. We took this as an opportunity to embed various suitable mat(h)erials in the textbook. It was also a great opportunity to engage deeply with the content of classes 1-3 and to integrate all the best pedagogical practices that we were aware of.

Many organizations and people have over time contributed to primary mathematics in India. However, those in the northern part of the country had not collaborated much with those working in the south. The latter group had inherited the works of PK Srinivasan. We had the vantage point of rooting our textbook endeavor to both sources. These were our dreams and it demanded quite a lot.

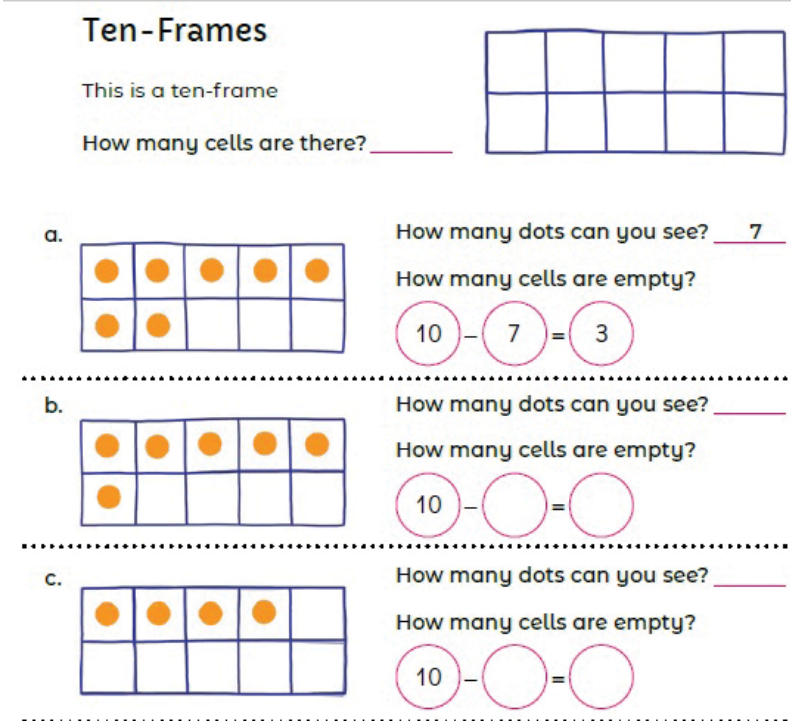
Since we were meeting only once a week every month, a lot of work had to happen beyond those meetings and had to be shared among the group. So, all the authors got comfortable with their laptops and in fact one of them even bought one! Till then, the smartphone had been their main gadget. WhatsApp was their main way of connecting with each other. However, soon everyone was sharing files over email, teleconferencing over phone and also skyping across the globe. Dr. Wagner was with us for some of the monthly meetings and kept in touch over email and Skype. During one such meeting, we visited a school to deepen our understanding of the ground realities and to get the perspectives of the teachers. Needless to say, the mountainous terrain of Sikkim did get in the way of connectivity, but could not dampen our spirit, or the pace of the work. Each chapter was reviewed and improved multiple times. The number of iterations reduced as we got more comfortable with the task.

The entire project was further enriched as we were joined by a group of designers from the state. Thanks to their efforts, the textbooks turned out to be colorful representing the local people and their culture. Because of some administrative hurdles, the work began a bit late. So, we faced a last minute rush in finalizing the textbooks. But, the hurdles faded when we saw the final version – it was all worth it!

Most of the authors had a good command over math. But it was not always easy to come up with stories and examples rooted in local contexts with a flavour of ESD. Each example had to be thought through. Each topic had to be introduced with the need of connecting the concept to the life of the people in Sikkim. The second challenge was to make the pedagogy constructivist – bringing in a flavour where children are encouraged to discover rather than the textbook (or an adult) telling them what to do. So, conversations between and among children as well as those with teachers and other adults were used often. Many a time, the initiating question would be posed by a child and the adult attempted to help the child (or children) figure things out rather than give a direct answer. Games and activities were also included wherever possible. Many of these were done as stories, so that the teacher (and the children) could get a sense of how to play.

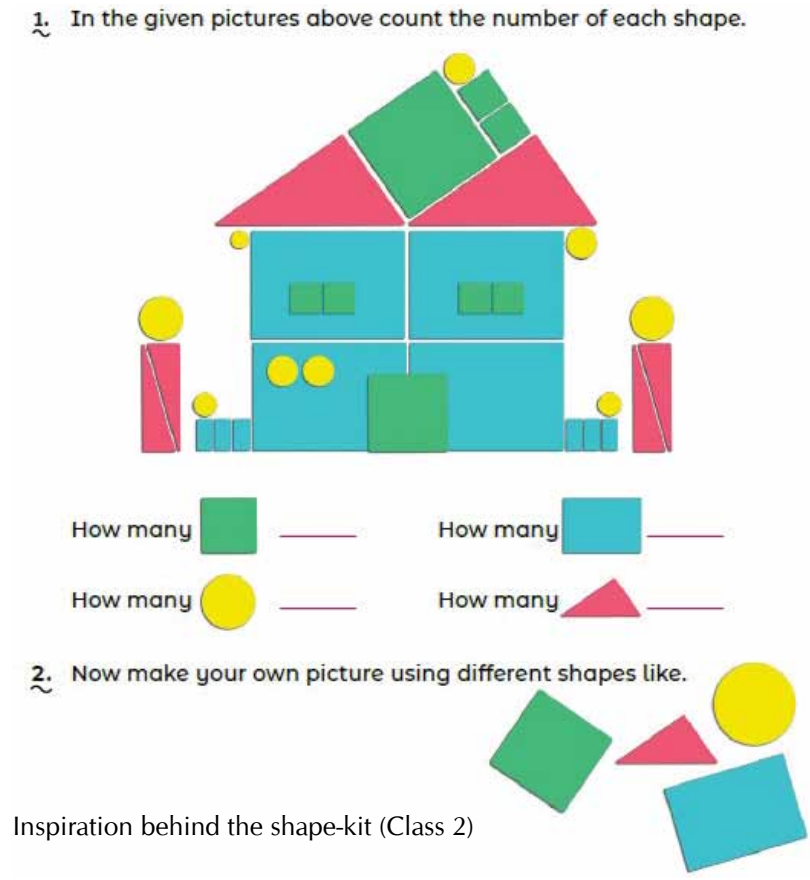
The textbooks were based on the NCERT books. But they were modified so that the local food, fruits, trees, games, festivals and various other aspects of the Sikkim culture were represented in the chapters. In addition, mat(h)erials were included to explain concepts. Since many of the authors had worked with Jodo Gyan, this part was relatively easy. All of them felt the need for mat(h)erials and understood the benefits they offer. We borrowed from the works of Maria Montessori, Rohit Dhankar (Digantar), Jodo Gyan and the pullouts (part of the magazine At Right Angles published by the Azim Premji University) by Padmapriya Shirali, a student of PK Srinivasan.

We added thematic chapters in both classes 2 and 3 to showcase how different mathematical concepts could be used in day-to-day life. It should be noted that the first thematic chapter appears only in class 4 of the NCERT textbooks. ESD was new for all of us. Almost all exemplars in the MGIEP guidebook involved higher level mathematics – using algebra and functions to model and optimize. So, we


Ten-Frames
This is a ten-frame
How many cells are there? _____
a.
How many dots can you see? 7
How many cells are empty?
10 – 7 = 3
b.
How many dots can you see? _____
How many cells are empty?
10 – ◯ = ◯
c.
How many dots can you see? _____
How many cells are empty?
10 – ◯ = ◯
Introduction of ten-frames (Class 1)

had to figure out what could be done with the basic concepts in the beginning years of primary mathematics. Apart from going local, we focused on fruits and vegetables rather than candies, rural situations in addition to urban ones, local measures used for weight and capacity and issues (like landslide) that people commonly face in Sikkim. Moreover, we included choices people make based on mathematics – e.g., why wheels are round or what would happen if the football was a cube.

For example, we decided to include a child with one leg in one chapter of class 2 to fulfill a mathematical need as well as to be inclusive. Another author liked the idea and added a child with one arm as a character, who is the most sought after player for a local game, in a chapter


1. In the given pictures above count the number of each shape.
How many
How many
How many
How many
2. Now make your own picture using different shapes like.

Inspiration behind the shape-kit (Class 2)

of class 1. We added Braille in the patterns chapter in class 3. In this manner we included different kinds of disability in each textbook.

We were aware that these textbooks would be very different from the existing ones and therefore the teachers would need a lot of hand holding. We decided to address this in two ways – (i) include "Teacher Note" wherever needed within the chapters, and (ii) add "Teacher Pages" for selected chapters in each textbook. In addition, we added a "Note for Parents and Teachers" at the beginning of each textbook to help them understand the thinking behind the approach. The learning outcomes for the respective class were also added to each textbook. Sikkim also got them reviewed by experts from the

Make your own mat(h)erials

The modified shape kit: It has four types of shapes with the given sizes (and the number of each type of piece).
Yellow circles with diameters: ~5cm (12), ~1.5cm (12), ~1cm (33)
Green squares with sides: 4cm (9), 2.5cm (18), 1cm (36)
Blue rectangles with sides: 3cm × 5cm (8), 2cm × 3cm (8), 1cm × 4cm (14)
Red right triangles with legs: 4cm-4cm (8), 2cm-2cm (16), 3cm-4cm (20), 2cm-5cm (24)

These can be made from any thick card type material including boxes and coloured paper. Bottle caps and toothpaste caps can be used to draw the circles.

With these one can make all possible types of triangles except the equilateral and all types of quads in addition to various explorations involving seriation, sorting, patterns and algebra, tiling, perimeter and area, and symmetry. They can foster creativity as well as provide various opportunities to think logically and justify.

For more such ideas on creating your own teaching mat(h)erials contact the author.

state. Their feedback was also incorporated.

After one year of hard work, we had an orientation workshop with the teachers who would use these new textbooks as a pilot. They went through various selected portions of the textbook, many activities and uses of various mat(h)erial embedded therein. They agreed that the new textbooks were very different from what they were using so far, and that it would require them to work hard. However, they were all willing to put in the effort. And they did! We set up a WhatsApp group with the teachers who tried out the new textbooks. They shared pictures and videos of the work children had done, the games they played in class, the mat(h)erials they made and used – the fun they had! Teachers saw children engaged in math activities even during break! In addition, they helped us identify many typos and other mistakes which we rectified and asked for clarifications as and when they needed.

Math Space also got enriched in the process. We had worked on mat(h)erials for numbers, algebra, geometry and mensuration. This project expanded that work to (i) measurements – length, weight and capacity, (ii) patterns, including tiling, and we also developed mat(h) erials from things around us, things we usually throw away. We developed a kit of triangles, circles and rectangles (including squares) to supplement the textbook. Later, it was modified keeping the needs of upper primary in mind. It turned out much richer than anticipated and continued to be useful for primary as well!

By the end of it all, we were enriched by the experience of working deeply with primary mathematics, had the privilege of building on the work of many others, stretched our horizons with regard to ESD and most importantly contributed meaningfully to the lives of the schoolchildren of an entire state.

[The views expressed here are those of the author and not necessarily the official views of Azim Premji University.]

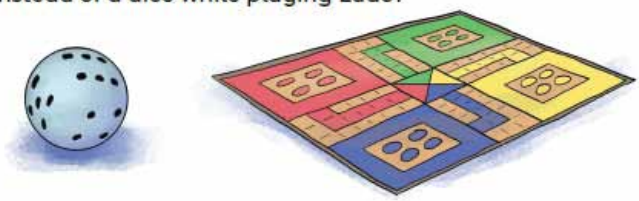
TP

Let us think

1. What would happen if your football was a big cube?

2. What is the shape of the wheel of a car? Can we have triangular wheels? Why?

3. What would happen if a marble (tol) is used instead of a dice while playing Ludo?



Choice of shapes impacting functionality (Class 3)

The multiple challenges of math

Sudha Ganesh Chella and Anuradha Jaishankar

Sudha is a post graduate in child development and family relations and a BEd in special education. She founded Diksa, a centre for children with learning difficulties in the year 2000. Sudha was recently honoured by the Rotary Club of Madras North with the Rotnorth Aasan Award for her contribution to the field. She was also honoured by CHILD – Centre for Holistic Integrated Learning and Development for her outstanding service and dedication towards individuals with learning difficulties. She can be reached at <sudhaganeshchella@gmail.com>.

Anuradha Jaishankar is a Special Educator working with Diksa for the last 12 years, teaching children with dyslexia. She is an Engineer and a Post Graduate in Childcare and Education. She specializes in teaching math to children with learning difficulties and life skill math to young adults. She has completed standard levels 1 and 2 of Instrumental Enrichment program from ICELP, Jerusalem. She has participated in several International Conferences on learning difficulties and primary math workshop conducted by the University of Cambridge. She plays a key role in conducting the "Grow with Math" program of Diksa. She also trains teachers on effective teaching of primary math. She can be reached at <vanujai@yahoo.com>.

Mathematics has a rich history of being considered the most difficult subject in school. So much so that many people consider it a “badge of honour” to say that they were bad in mathematics when in school. These difficulties were generally attributed to a lack of intelligence or hard work.

However, in the past few decades, there has been a lot of research into understanding the reasons behind these difficulties. Three major types of difficulties have been recognized; those related to the nature of mathematics as a discipline, those related to the way it is being taught and others related to the learner herself. It has also been recognized that all three are related.

Some learners have certain learning difficulties possibly related to genetic or brain structures. The underlying causes for many of these conditions are still not fully known. In most cases there is no effect on general mental functions.

The most commonly encountered conditions are dyslexia and dyscalculia. Dyslexia relates to difficulties in reading and writing and dyscalculia to difficulties with mathematical concepts and ways of thinking.

We would like to clarify here that the term “mathematical ways of thinking” implies far more than the concepts and skills that we learn in mathematics in school. It involves a wide range of cognitive skills for understanding and transacting any body of knowledge.

Though dyscalculia may not affect language learning in a significant way, the reverse is not true. This is because, ultimately even mathematics has to be understood and expressed through the medium of language. The other learning difficulties namely dysgraphia (issues in handwriting), dyssemia (difficulty in spatial orientation), dyspraxia (difficulty in motor

coordination), dyslalia (issues in articulation), and dysphasia (difficulty with language) also have a significant impact on a child's overall learning ability.

In this article, we will try and describe some of the difficulties such students face in learning mathematics under three broad headings – Basic developmental deficits, Perceptual difficulties and Cognitive deficits.

Basic developmental deficits

Developmental delay

Children with physical and motor developmental delays face gross and fine motor issues. When a child has difficulty in stringing beads or in jumping, she misses the early math learning experiences that happen at home during the initial years of 1 to 3.

Sensory integration issues

Lack of synchronicity in functioning of various senses leads to sensory integration issues and also impairs brain-body connections. Such a child will have issues in eye-hand coordination and poor perceptual number sense. These result in delay in learning pre-number concepts like more/less, big/small, colours, etc.

Speech delay

A child picks up many of the pre-number concepts such as one/many, long/short at home during the early years. Delay in speech impacts the early expressive language of math. They don't get to speak and practice words like more, less, one, two, etc.

Attention

A child finds it difficult to stay on task either due to attention deficit hyperactive disorder (ADHD) or attention deficit disorder (ADD). In either case, lack of attention affects concentration and thus impairs learning in all aspects.

Memory

Good memory is essential for learning, retention and retrieval of concepts, facts and skills. Children may have difficulties in both short-term memory (remembering a number during carry over addition, copying a number from the board) and long-term memory (recalling a formula or a multiplication fact). Difficulty in recalling math facts like tables, labels, procedures such as long division, multiplication, etc., are some of the difficulties faced by children with poor memory.

Counting

Though it sounds like an easy task, counting in various contexts requires specific cognitive skills. If the concepts of "one to one correspondence" is not internalized, she grows up to have difficulties in counting from memory, counting the sides of an object like cube, counting objects with increasing criteria (number of square pieces with green on top and blue at the bottom), counting objects in motion, counting while moving, etc. It later poses difficulties in skip counting, computation of numbers, handling money and time, etc.

Letter-sound association

A child with difficulty in letter-sound association can find it hard to correspond a numeral to its quantity. This manifests as lack of number sense, inability to compare quantities and

Illustrations: Swetha G Nambiar

understand basic operations, reading a word problem and writing statements.

Language

A child with poor language skills will not be able to comprehend commonly used words in a math context (balance, appreciate, etc.), math specific vocabulary (perimeter, hypotenuse) and word problems.

Perceptual difficulties

Spatial orientation

A good spatial orientation ability helps children perceive sides (front, back, left, right) and directions. Clarity on direction and position is important to understand topics like place value, basic operations, graph and geometry. Lack of this skill makes it difficult to learn place value, ascending and descending order, understand the characteristics of a shape irrespective of its orientation, addition and subtraction with regrouping, signs, etc.

Temporal sense

Time has always been a difficult concept for young children. If the child is unable to perceive the idea of time, she does a poor planning of time (always in a hurry or too slow), struggles to follow the units of time like minutes, hours, days of the week, months of the year, concepts like before and after.

Lack of closure

The idea of part-whole relationship is an important concept to comprehend topics such as shapes, graphs, geometry, mensuration, etc. The child who has lack of closure, sees the parts of a whole as independent entities and misses the connection between them. All diagrammatic representations become a challenge for him.

Visualizing math

Using simple manipulatives – place value concept can be mirrored with ice cream sticks. A set of 10 sticks bundled using a rubber band will mirror the idea of tens. So 32 can be represented with 3 bundles and 2 sticks, whereas 23 can be represented with 2 bundles and 3 sticks. It also visually shows the student that 32 is bigger than 23. It does not require a rule to be memorized.

Personal life experiences – Our life is full of events which mirror the four arithmetic operations – addition, subtraction, division and multiplication. On your birthday, getting two gifts from your school friends and three gifts from your relatives, mirrors the idea of addition. You have a total of five gifts and can be written as $2+3=5$.

Role Play – Imagine an activity when one student is asked to take 12 crayons and distribute them equally to four students.

At the end of above exercise, he will realize that he has no crayons with him and also that each student has got three crayons.

This mirrors the idea of division and can be written as $12 \div 4 = 3$.

Identification of characteristics of different shapes

The ability to identify and discriminate each shape/size is a prerequisite to learn geometry. When a child is unable to internalize the characteristics of a figure properly, he will have issues in identifying shapes and angles based on inter-class and intraclass differences.

Classification

If a child is unable to sort and group things based on specific criteria, she faces difficulties in understanding properties of numbers and their relations (odd and even numbers, factors, multiples), identifying shapes, mensuration (area, perimeter), etc.

Auditory-visual integration

A child who struggles to match oral instructions with visual content finds it hard to follow instructions and copy from the board. This can severely affect classroom learning.

Cognitive deficits

There are cognitive deficits which affect learning in a general way – Information gathering, information processing and information output.

Information gathering – lack of systematic information gathering, inability to filter the relevant from the irrelevant, poor attention and improper perception.

Information processing – comprehension, problem definition, sequencing, comparing, hypothetical thinking, extrapolation, estimation, logical thinking, analysis and synthesis.

Information output – Improper geometry work, illogical answers due to impulsivity, not writing the units, statements, incomplete

work, lack of details, inability to communicate effectively.

Educating children with learning difficulties

Educating children with learning difficulties in integrated classes can be a challenging experience for the teachers. These students will need to be taught differently and require time to comprehend, assimilate and internalize concepts. Revisiting the concepts regularly helps them.

But teachers need to remember that these learning difficulties get amplified when such children are stressed or under pressure to perform. Unconventional assessments are better tools to understand such students' conceptual clarity and performance.

Children with dyscalculia may not have all the issues listed above. Also, dyscalculia can occur in isolation. It may or may not happen along with dyslexia, ADHD, etc.

Grow with math program

Diksa Learning Centre has been running a program, "Grow With Math", for the last three years, combining our understanding of learning difficulties with multi-sensory ways of learning math. Some of the highlights of the program are

a. Mirroring primary math concepts through simple manipulatives, personal life experiences and role play (see box for how you can do this in your classroom).
b. Visualization of numbers, number properties and number relations.
c. Structured, sequential and interactive sessions with a spiral curriculum.
d. "Math Chat" time which encourages students to share their math experiences in daily life and develop their language ability.
e. Multi-sensory learning through visual, auditory and kinaesthetic activities.
f. Creatively designed activity sheets.
g. Activities and activity sheets that aim at developing logical reasoning and inculcating mathematical thinking in students.
h. Peer interaction and peer learning.
i. Projects involving life skills math.

The program has helped children overcome their math phobia and instilled willingness to learn and experience mathematics. Regular exposure to real life math situations helps them see the connection between learning and experience. Students are willing to talk 'math' in the classroom. They are keen to try new and different methods. They are able to 'mathematize' simple problems. Overall, there is a tremendous increase in motivation and understanding in the students.

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Math for the mind's eye

Kalyani C

The author has been a volunteer for blind students for more than 25 years now. All her students are well placed and leading dignified and independent lives. These days she converts print material into audio for blind teachers in Tamil Nadu. She can be reached at <kalyanimouli@gmail.com>.

"Mathematics is not about numbers, equations, computations or algorithms; it is about understanding."
– **William Paul Thurston** (1946 – 2012), American mathematician.

Vision is fundamental to learning. Sighted people obtain information through their vision, especially through their observational skills. Such incidental learning does not happen for non-sighted people. Mathematics is abstract. One of the ways of understanding abstraction is to indirectly mirror them with physical objects and visualize the images. Non-sighted persons are denied of this learning opportunity, which makes learning mathematics a great challenge. Not only math, acquiring basic literacy itself is a great challenge for them. Print material is of no use. They need all print material in the Braille code. Braille is a tactile reading and writing system for the blind. To achieve a literacy level on par with their sighted peers, visually impaired students need to have adequate supply of Braille textbooks. Geometry and spatial sense can be taught through concrete hands-on experiences.

The blind child needs training in using his hands as his 'eyes' and this training should start from early childhood at home. The child learns to identify his mother by touch. Their major channels of information are auditory, tactile and kinesthetic. Most of the challenges in learning math can be overcome by teaching them in an appropriate manner. Mathematics can be taught using a combination of tools like the Braille code, Abacus, tactile materials and the Taylor Frame along with the Nemeth Code. Similarly, mathematical concepts can be taught using various geometric shapes of different sizes and textures, felt boards and symbol stickers. Over and above all these, a teacher well-trained in teaching math using all these special devices is a must.

Assistive technology has removed many barriers to education, especially in the understanding of math for visually impaired individuals. A short description of some of these technologies with illustrations is provided for reference in the annex to this article.

Special schools and integrated schools

Children with visual impairment go either to integrated schools or to special schools for the blind. Special schools are schools that are

Braille Slate and Stylus

Braille reading

Braille Writing

Braille type writer

specifically designed, staffed and resourced to provide appropriate education to special children. They are often better equipped to provide individualized education, addressing the specific needs of these children.

Integrated education is not simply about placing a blind child in a regular classroom. The child needs assistance. The most important factor in the education of visually impaired students is the resource teacher. Resource teachers are trained to teach Braille literacy and mathematics using the Taylor Frame along with the Nemeth Code and tactile graphics. In the integrated system of education, a resource teacher is appointed along with regular classroom teachers to take care of the education of the blind children. It is their responsibility to train the students in the plus curriculum as well. Plus curriculum includes, among other things, Braille literacy, learning math using abacus, and Taylor Frame, orientation and mobility training and daily-living skills. Unless these skills are learned and mastered, visually impaired individuals are at high risk of leading a lonely, isolated, unproductive life. The general classroom teacher and the resource teacher have to work together to successfully teach the visually impaired child.

Importance of elementary education

Elementary education in special schools is very important for blind children. It gives a strong foundation for their future learning in the sighted world along with sighted peers. It helps build their self-esteem, self-confidence and prepares them to face their challenges boldly. Whereas, elementary education in integrated schools shatters their aspirations; makes them diffident. Due to shortage of study materials, lack of awareness about their abilities and a lack of empathy from fellow students they are not able to acquire even the basic literacy and numeracy skills. Helping blind students understand their strengths and weaknesses is important while teaching them math.

Situation in our education departments

For the education system to work effectively, particularly for children with special needs, several deep changes are required. The most important is an attitudinal change among administrators in our education departments. MHRD – Ministry of Human Resource Development – should bring education of all children, both non-disabled and disabled under its wing. At present, special education is under the Ministry of Social Justice and Empowerment (MOSJE) while mainstream education comes under MHRD. This creates dichotomy and fragmentation.

The status of most integrated schools in Tamil Nadu which enroll blind children is not satisfactory. There is always a shortage of Braille textbooks as well as resource teachers. Even if the books are available, teachers do not know how to read in Braille nor do they know how to teach mathematics using the Taylor Frame and abacus. To rectify this problem, the government needs to recruit enough resource teachers and give them in-service training.

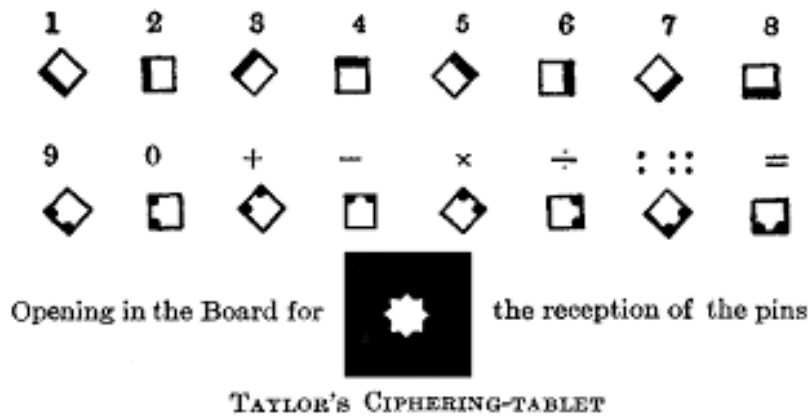
Visually impaired students in Tamil Nadu learn mathematics only up to class X. They cannot pursue it in their higher studies, not because of a lack of ability, but because there are no teachers to teach them mathematics at that level.

Some findings from the field

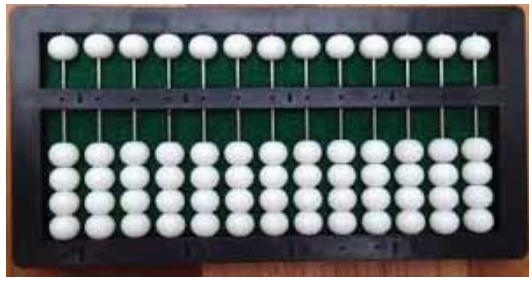
A blind research scholar undertook a study to evaluate the status of education of the blind in the schools of Tamil Nadu.

The research study reveals the following ground realities –

a) Resource teachers are not government employees. They are appointed by trusts or non-governmental agencies.

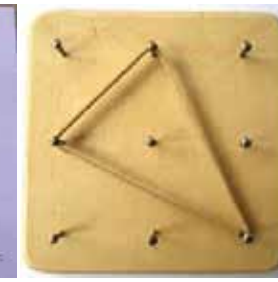


Nemeth Coded Types



Cranmar abacus

Tactile graphics



Geo board

They do not receive their salaries on regular basis from these agencies. Due to this problem they are forced to leave the job. Hence, there is always a shortage of resource teachers.

b) Seven out of the 17 integrated schools in Tamil Nadu did not have resource teachers.

c) Braille textbooks were always in short supply and none of the blind students had their own copies of Braille textbooks to read and practice. They were not provided with tactile maps or tactile graphics for better understanding of concepts. In some cases, though they had Taylor Frames, there was no teacher to teach them mathematics using Taylor Frames or abacus. As the teachers were not trained in Braille, student assessment was done only on their oral skills. Students were not given any written assignment as homework or encouraged to write their answers (in Braille) during examinations.

d) In special schools for the blind, students are taught compensatory skills right from class I, whereas in integrated schools it depends on the availability of resource teachers. Hence, most parents preferred sending their blind children to special schools rather than to integrated schools.

Some inspiring stories

In spite of these hurdles, there are students who successfully overcome difficulties and accomplish just about anything their sighted peers can. Some of them have even proved that they can be leaders and that they can help other disabled people in leading a normal dignified life. To name a few –

Beno Zephine, Indian Diplomat, is the first 100% blind Indian Foreign Service Officer.

Arunadevi, government employee, has done her BA, MSW and M Phil in sociology. Along with her friends, who are also disabled, she has formed an organization named, Society for the Rights of All Women with Disabilities and works for the empowerment of disabled destitute women in Tamil Nadu.

Saraswathi, PhD (English Literature) is a teacher in a regular government school in Tamil Nadu teaching English to sighted students up to class 10. She is happily married and has a child.

The way ahead

I conclude by summarizing the most important areas of education, including mathematics education, for the blind which needs to be improved.

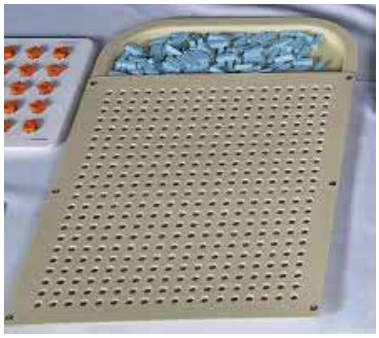
1. Training for all teachers to understand the emotional and learning needs of blind children.
2. Adequate resource teachers trained in Braille reading and in using assistive technologies.
3. Strengthening effective elementary education for blind students in all schools.
4. Availability of Braille textbooks for students.

Annex

Assistive technologies

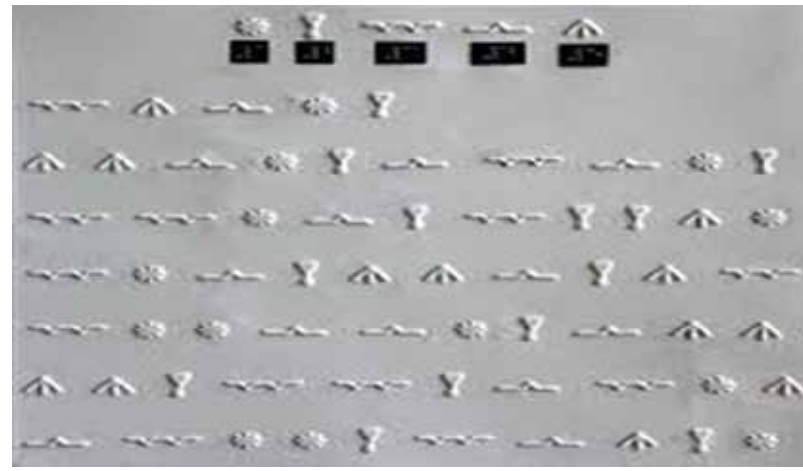
Assistive devices essential for learning compensatory skills are as follows:

- *Braille* – Braille is a code, a system of six dots that represent the letters of the alphabet. Braille offers distinctive advantages for children who want to learn the fundamentals of language, such as spelling and phonetics.
- *Braille Slate* – Braille Slate is used in learning Braille dots using the touch and feel method.
- *Stylus* – Stylus is used in marking the Braille dots on paper.
- *Taylor Frame* – Taylor's mathematical frame is used to solve mathematical problems and calculations. It is suitable for teaching arithmetic to visually impaired persons.

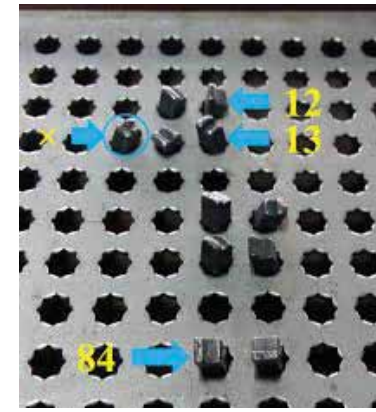

Taylor Frame and Types / Pegs

The Frame has star shaped holes with eight angles, thus allowing double-ended metal types or pegs to be placed in different positions according to a set system. Types or pegs are used to write numbers on the Frame. All basic arithmetic operations can be carried out on the Frame.

- *Nemeth Code* – Nemeth Code is a special type of Braille used for mathematics and science notations. It was developed in 1946 by Dr Abraham Nemeth (1918 – 2013), a Professor of mathematics. With the Nemeth Code, one can render all mathematical and technical documents into the six-dot Braille. The Nemeth Code has been instrumental in opening up careers in science and mathematics to people who are blind or have other visual impairments.


Arithmetic Sample

- *Cranmar Abacus* – Abacus for the blind students is called Cranmar Abacus. It is a tool used for mathematical calculations. This can also help in one's daily life, for instance calculating prices while buying groceries or shopping. This is a modified form of the Japanese Abacus, Soroban.
- *Geometry Kit* – The Geometry Kit, specially designed for visually impaired children enables them to understand geometrical concepts clearly.
- *Tactile graphics* – Tactile graphics, including tactile maps and tactile graphs are images that use raised surfaces so that visually impaired persons can feel them with their sensitive fingers.



Taylor Arithmetic Frame

- *Screen Reading Software*, especially JAWS (Job Access With Speech) and NVDA (Non-Visual Desktop Access), enables them to read all the information available on the internet and acquire knowledge.

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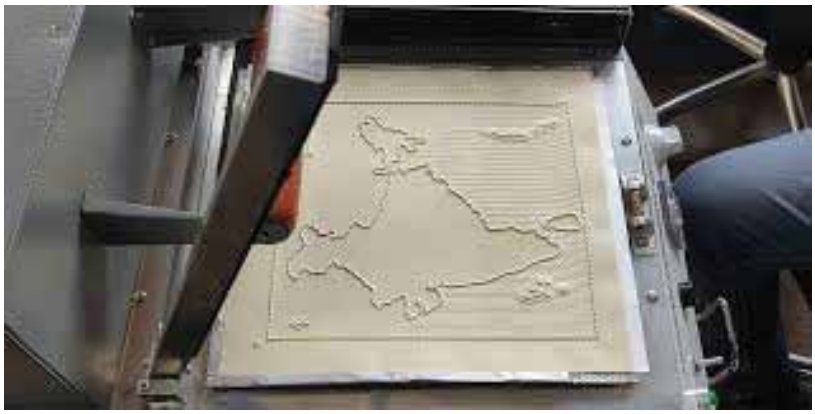

India map tactile

Moving from fear to fascination

Umamaheswari Subramanian

The author is a passionate educator currently serving in a public school in the US. She believes that anyone can learn and excel in mathematics and that educators have the responsibility to tap students' curiosity to facilitate learning mathematics with joy. She can be reached at <umaeducator@gmail.com>.

As a mathematics teacher and educator, I have always enjoyed the privileges I have earned and wondered why mathematics continues to be a phobia amongst the common public. While there are several commonalities in how people in India and the United States view mathematics, there are some remarkable differences in the way mathematics is taught in these two countries. I have taught mathematics at all levels for more than a decade in India and for nearly two decades in an urban city in the United States. In this article, I share some experiences that transformed my classroom into a more robust one with students exploring and engaging in the learning process.

The math classroom

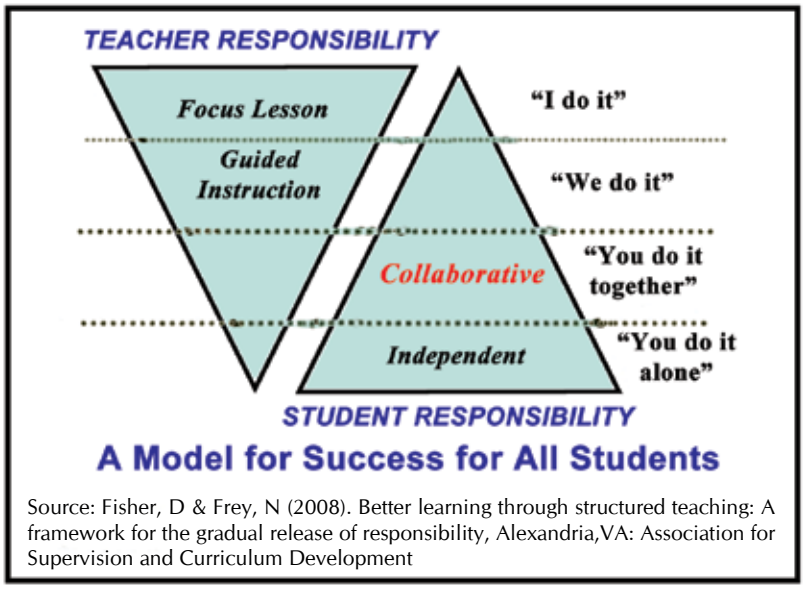
As a school student and as a high school teacher in India, I was accustomed to the teacher giving all the information and the students passively listening and simply imitating the teacher in solving problems. There was never a discussion in the class amongst students and rarely did students ask questions. The scenario changed when I started teaching in a public school in the US. I had to make several changes to my teaching style to succeed in my new surroundings. High school students in the US have the freedom and courage to speak their mind about the class and my students demanded that the class be more engaging. They wanted to know why they should learn maths and I began to start my classes talking about a real world problem that was directly or indirectly related to the math concept in their lesson. This triggered my students' desire to participate in the learning process.

As a new teacher in the US, I realized that I had to give my students a bait to draw them into learning the topic for the day. I began using short incomplete video clips followed by classroom discussions as part of my teaching routine. This is something I learnt from Dan Meyer (a math teacher) and his Three Act Tasks, which are interactive lessons that start with a video and can engage even the most unmotivated because of the various entry levels and the sense of achievement students derive by participating. Currently, the Three Act Tasks and its various modifications are widely used to engage students in math classrooms.

Gradual release of responsibilities

During the beginning of the 21st century, schools in the US introduced the Gradual Release of Responsibility as an instructional framework to transform students from being merely passive listeners to becoming active participants in their learning journey. Here, the teacher is no longer the sole authority in the classroom; the students take the ropes of learning in their own hands and the teacher is more of an observer.

Adapting to my new responsibilities as a teacher, I started assigning collaborative assignments to my students. In this process I also had to learn the strengths of each student so that I could make



Source: Fisher, D & Frey, N (2008). Better learning through structured teaching: A framework for the gradual release of responsibility, Alexandria,VA: Association for Supervision and Curriculum Development

flexible groups where every student could contribute. I also had to provide scaffolds to those groups I knew would need more support.

Creativity in mathematics

What has creativity got to do with mathematics? Mathematics in the Indian classrooms is always taught as a study of formulas and their direct application. Teachers in the US do things a little differently. They assess their students' conceptual knowledge using mathematical projects and games. BINGO and Jeopardy are often used in the math classroom for review before an assessment.

Research indicates that students learn and retain concepts when the learning concepts are relevant to them. My fellow math teachers and I created a project to help students understand the concept of a 'function', its domain and range and the key characteristics of the graph using their real-life experiences. X represented the age of the student and f(x) represented the student's level of happiness expressed as a number from 1 to 10, with 1 representing the unhappiest year and 10 representing a great year. (For example, when I was 9 years of age, I was made class president and all my teachers loved me. This was my best year and I will give it a score of 10. The next year, I lost my grandmother and I was very sad and so will give it a 2). Students were happy to share their life experiences in the form of a function and could describe intervals of increase/decrease and other characteristics of their function. Another project that I particularly enjoyed, required students to design a logo using various families of functions in a restricted domain and draw the same on a graphing calculator.

Teacher certification and renewal

Anyone wanting to teach in any school in the US is required to have a license from the state. The requirements are different for different states, but all involve completing a teacher education program after finishing a Bachelor's degree and taking an assessment. The certification process is meant to ensure that teachers meet certain standards in the areas of pedagogy and the subject area they plan to teach in addition to clearing a background check.

An interesting and noteworthy fact is that the certification is valid only for a period of three to five years and one must keep renewing the certificate in order to continue teaching. Teachers are also required to complete certain hours of professional learning related to their subject and pedagogy through workshops, seminars and conferences to renew their certification. This ensures that teachers are constantly updating their knowledge with the latest information in not only pedagogy, but also technology and their subject area. Also, there are several teacher conferences, workshops conducted by various colleges of education and universities throughout the year and during summer. These conferences allow practicing teachers to share best practices, innovations in their own classrooms, technology aided assignments, games and projects related to concepts for each grade level; at the same time, teachers also gain confidence and feel appreciated when sharing their best practices with their peers from other parts of the country. Some universities also provide opportunities for teachers (with a stipend) to come and work in their research department as a way of learning about the latest research in the various fields of science.

Educational publications, coaching and tutoring institutes and other industrial organizations in India will have to come forward to hold teacher conferences for the professional development of teachers. Educational trusts and institutions running schools should also conduct research on teaching practices and student

learning and should encourage their teachers to share their best practices in order to learn from one another.

Gifted and talented

Have you had students in your class who always have the answers to everything you ask even before you teach the lesson? In India, such students can become teachers' assistants helping the teacher in different ways. What do the students gain by being the teacher's assistant? In some rare cases, if the teacher has had exposure to different mathematical competitions, he/she may encourage the student to compete as well. In the US, however, law mandates that gifted and talented students be identified based on teacher/parent recommendations by using multiple criteria (mental ability, achievement, creativity, and motivation). Students thus identified will attend the Gifted and Talented program, where students may be placed in an honours class that follows a differentiated rigorous curriculum or in an accelerated curriculum completing two years of work in one year thus allowing the gifted students to move ahead of their peers.

At the high school level, the gifted students can complete all their coursework much earlier when they are in their 11th or 12th grade and can take a few college level classes in school (advanced placement classes offered by the college board) and obtain college credit or they can be enrolled in college while still being a senior in high school. I am looking forward to the days when Indian teachers will also start nurturing gifted and talented students and providing them with opportunities to accelerate their learning.

Mathematics continues to be a gatekeeper to higher education everywhere and even today we have high school students who either hate or fear math. This is true even amongst students in the United States in spite of the ever changing research-based teaching practices there. Mathematics teachers and educators everywhere have a huge role to play to make mathematics a learner friendly subject and to remove math phobia in learners. This starts with teachers making changes in the classroom; shift from teaching math as a set of rules and formulas to creating meaningful, relevant and challenging learning experiences for students. The education system has a more important role in terms of designing the curriculum, training teachers, conducting research on students' learning; but all this will only start with teachers demanding the same from the system.

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Maths education in India: policy and reality

Rajesh Kumar Thakur

The author is a mathematics teacher in Delhi Government School, Sector - 16, Rohini. He has written 58 books, 62 ebooks, and 500 articles for different magazines and newspapers. He can be reached at <rkthakur1974@gmail.com>. You can also tweet him at @R_K_THAKUR

Mathematics, the synonym for phobia, is defined as the *science of space, numbers and quantity* by the Oxford English Dictionary. Mathematics is called the queen of all subjects and also the mother of all sciences. The foundation of a society is mathematically inclined and the importance of mathematics can be gauged from what Roger Bacon (1214-1294), English philosopher and scholar once said, *"Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or things of the world."*

India has displayed its love for mathematics since the Vedic period. The Rig Veda talks about a huge number like the *parardha* (10^{12}); the book Lalit Vistara written on the life of the Buddha speaks about *tallakshana*, another large number (10^{53}); the Valmiki Ramayana speaks about yet another big number, *mahaugh* (10^{60}), in Yudha Kanda. The book written by Satpathi Brahmins and Sulvasutra written during the 8th century BC talk about deep geometric concepts. Mathematics was alive and thriving between the 4th and 12th century AD under Aryabhata, Brahmagupta and Bhaskaracharya. India's contribution to the world of mathematics includes the discovery of zero, the decimal number system and negative numbers, and yet today India's presence in the field of mathematics is almost negligible.

So, what went wrong with India's mathematical dreams? Why are we not producing good mathematicians? According to the 2014 Field Medal winner and member of NEP 2019, Manjula Bhargava, *"The biggest mistake that happened in India after independence was the separation of teaching and research in India's higher education system. India has some fantastic research institutions like Tata Institute of Fundamental Research, International Centre for Theoretical Sciences, Indian Statistical Institute but they are little islands of excellence. And then there is the whole teaching sector, state universities where no research happens."*

The National Achievement Survey 2017, conducted by the MHRD, to assess learning outcomes among children across India in different subjects shows class 3, 4 and 5 students getting 64%, 53% and 42% respectively in the mathematics test. This shows a trend of declining interest in mathematics. The situation is even more alarming at the higher level as the Central Board of Secondary Education (CBSE) confirms that only 21% students passing class 10 opt for mathematics

at the senior secondary level. Why? From a country that had a rich mathematical tradition, how did we turn into a math-fearing nation? Have we not made efforts to bring back the popularity of maths?

As early as 1968 The Kothari Commission had placed emphasis on the importance of mathematics. It said, *"With a view to accelerating the growth of the national economy, science education and research should receive high priority. Science and mathematics should be an integral part of general education till the end of the school stage."*

The commission had suggested the introduction of mathematics at two levels. General mathematics was made compulsory up to class 10 and advanced mathematics was introduced at the secondary level as an optional subject. General mathematics comprises arithmetic, algebra and geometry, whereas advanced mathematics comprises integers, quadratic equation, logarithm and coordinate geometry. But the system failed in the implementation of this change. The government neither trained nor could recruit good teachers of the subject.

Then came the National Policy on Education 1986. Though the NPE was a visionary educational policy, it too failed to bring about significant changes to math education in India. Unattractive classroom environment, teacher absenteeism, teacher-centred pedagogy and low skill attainment were some of the reasons. In 1992, the NPE was modified to once again bring in the two level mathematics (with some changes) as suggested by the Kothari Commission in 1968. The curriculum was redesigned to suit modern day needs and trained teachers were appointed to teach the subject, mathematics kits were supplied under Operation Black Board. But due to a vast syllabus, non-availability of good textbooks and no systems in place for the professional upgradation of teachers, this policy like the ones before it failed.

India has so far had all well-thought out educational policies prepared by learned people, but they all fail miserably at one thing – seeking the views of the stakeholders – students, teachers, schools and parents.

In 1993 the government constituted a National Advisory Committee under Professor Yashpal to look into the issue of overburdening of schoolchildren. This committee rightly pointed out that the math taught in schools had no connection to children's lived reality and that this should be corrected if math was to become popular again. This report followed by NCF 2005 highlighted the need to reduce the size of the curriculum, ensure quality education for all and focus on activity-based teaching. The new syllabi proposed by NCF 2005 emphasized reason and conceptual understanding at every stage. Tackling mathematical anxiety was also addressed. According to NCF 2005, mathematics is more than formulas and mechanical procedures and an effort should be made to help children enjoy mathematics rather than fear it. The word *mathematization* was stressed upon. The policy document says – *"Developing children's abilities for mathematization is the main goal of mathematics education. The narrow aim of school mathematics is to develop 'useful' capabilities, particularly those relating to numeracy – numbers, number operations, measurement, decimals and percentages. The higher aim is to develop the child's resources to think and reason mathematically, to pursue assumptions to their logical conclusion and to handle abstraction. It includes a way of doing things, and the ability and the attitude to formulate and solve problems."* (NCERT, 2005, p. 42).

NCF 2005, NEP 2019 or the older NPE 1968 and 1986, were all well-thought out policy documents prepared by learned people, but all these policies failed miserably at one thing – seeking the views of the stakeholders – students, teachers, schools and parents. Only when those involved in the day-to-day transaction of education are consulted will we know whether the proposals put forth can be implemented on the ground successfully or not. What is the point of saying math teaching should be made fun and innovative when we don't have quality teachers who are passionate enough to uplift the standard of mathematics, when textbooks produced are bad and when we have not been able to link mathematics with daily life?

In the year 2012 when India was celebrating the 125th birth anniversary of Srinivasa Ramanujan, Prime Minister Manmohan Singh had said, *"It is a matter of concern that for a country of our size, the number of competent mathematicians we have is badly inadequate. Over more than the last three decades, many of our young men and women with natural ability in mathematics have not pursued the discipline at advanced*

levels. This resulted in a decline in the quality of math teachers in schools and colleges."

Let's explore the reasons for and solutions to the unpopularity of mathematics at the mass level.

1. *Poor quality of books:* Textbooks at the primary level should have children's favourite cartoon characters explaining the mathematical concepts. The link between math and real life should be made very apparent. Mathematical games, puzzles, titbits on the life of mathematicians, history of mathematics, new research and discovery in mathematics should all be an integral part of textbooks, which should of course be updated often.

2. *Faulty education policy:* Educational policies in India are designed by people who are unaware of the ground realities – school infrastructure, student-teacher ratio, the knowledge level of teachers, etc. Any education committee formed should have practicing teachers and other stakeholders as active members.

3. *Shortage of quality teachers:* Most students in rural area are first generation learners. They are completely dependent on their teachers for education, but there is a shortage of good trained teachers. We need teachers who can teach mathematics using technology, can relate mathematics with daily life. Mathematics teaching should not be monotonous. We need teachers to go beyond the textbook and not be in a hurry to complete the syllabus. It is not a teacher's job to tell her students how to score well in an exam, it is her job to show them how to apply their classroom knowledge to solve real life problems.

We have reputable institutions like the IIT, AIIMS, IIM, IIIT, which produce quality engineers, doctors and managers but in most of our universities we hardly focus on research in mathematics. The education system in India is inclined towards making money. A student who is good in mathematics prefers to go into engineering rather than pursue mathematical research.

4. *Lack of research institutes:* In the last 70 years of independence, we have built reputable institutions like the IIT, AIIMS, IIM, IIIT, which produce quality engineers, doctors and managers but in most of our universities we hardly focus on research in mathematics. Besides TIFR, IISc, ISI, IMS, etc. we don't have any institution that engages students in the field of research. The education system in India is inclined towards making money. A student who is good in mathematics prefers to go into engineering rather than pursue mathematical research.

5. *Lack of awareness of careers in mathematics:* In India, engineering or teaching are the two known career option for students of math. Different career options need to be explored. Actuary, data analysis, gaming, psychometry, reseach, operation research, financial analysis, cryptology are a few areas where people from mathematics background are in great demand. Even Facebook, Google hire people with a mathematics background.

Mathematics is not about calculations or getting jobs by cracking competitive exams, it is an integral part of our life. The policy formed by the government for the upliftment of mathematics should be implemented with proper planning and with full knowledge of ground realities. Math is a beautiful and exciting subject and unless we realize this we will not be able to bring back the glorious days of math.

TP

Folk mathematics

S N Gananath

The author has taught in Rishi Valley School in the past and is an independent consultant and teacher trainer now. He lives in Mysore and can be reached at <snganananth@gmail.com>.

The definition of a good mathematical problem is the mathematics it generates rather than the problem itself. – **Andrew Wiles** (English mathematician)

Long ago, there lived a family in a village and the father was on his death bed. He called his three sons and asked them to divide the horses he owned among themselves after his death. The eldest son gets 1/2 of them, the youngest son gets 1/9 and the one in between gets 1/3.

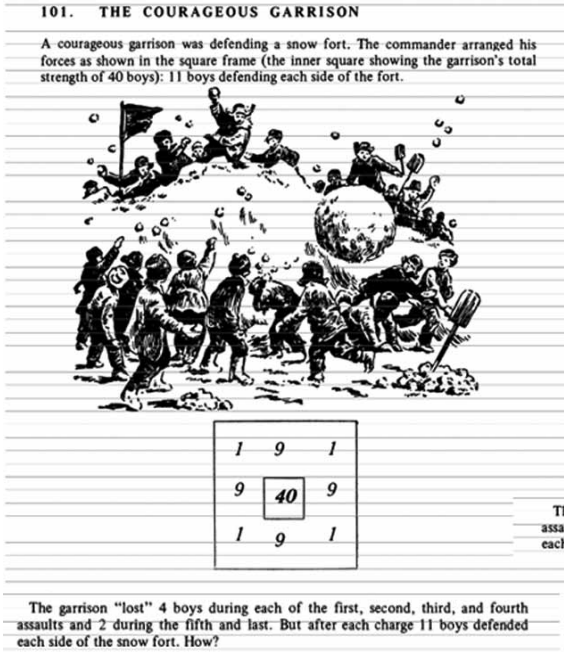
But this created a problem as there were 17 horses and naturally his children were confused. How do we divide 17 horses as per his wish without cutting them?

This is a folk mathematics problem and it is no surprise that this popular puzzle is prevalent in many parts of India and I have discovered it in Rajasthan, Bihar, Andhra Pradesh, etc., in my interactions with schoolteachers and even children. A closer examination reveals the enormous potential to convert such problems into a series of pedagogically sound and motivating activity sets. Some of the core objectives of mathematics teaching, viz., problem solving, creating extensions, problem posing, generalization, computational automaticity, spatial thinking and so on can be addressed by an imaginative and zealous teacher. This article intends to explore such a possibility and create appropriate pointers.

Let us consider another puzzle.

The courageous garrison

This appeared in the outstanding puzzle book *Moscow puzzles* by Boris A. Kordemsky, a book that has inspired a generation of mathematics enthusiasts.

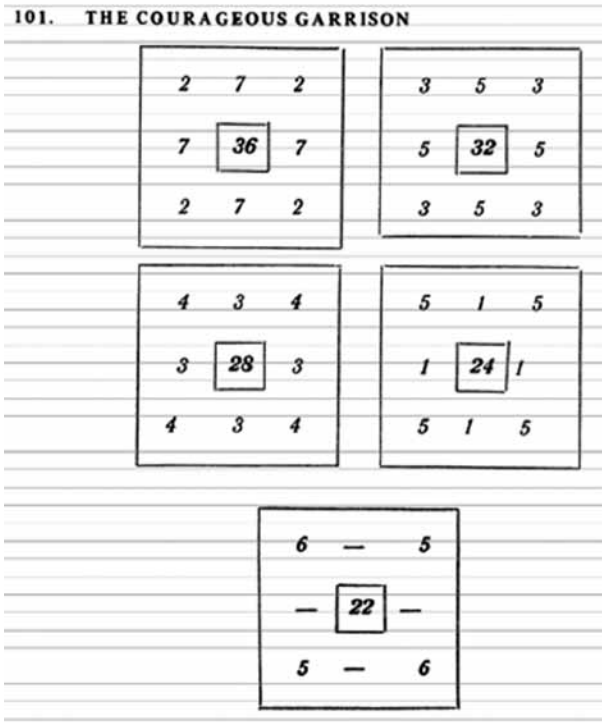


101. THE COURAGEOUS GARRISON

A courageous garrison was defending a snow fort. The commander arranged his forces as shown in the square frame (the inner square showing the garrison's total strength of 40 boys): 11 boys defending each side of the fort.

The garrison "lost" 4 boys during each of the first, second, third, and fourth assaults and 2 during the fifth and last. But after each charge 11 boys defended each side of the snow fort. How?

The solutions are here. You will notice the rearrangement of these boys after each assault.



Curiously, a puzzle very similar to this is popular in different parts of India. In that puzzle, a king has 32 horses and has arranged them in a square like this (see Fig 2) such that he can see 12 horses along each line of sight as indicated in the picture.

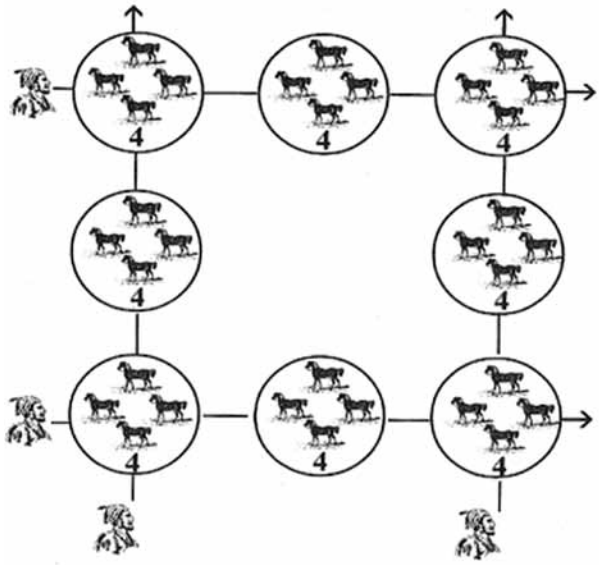
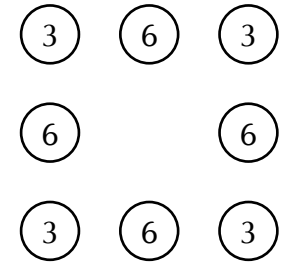


Fig 2

The king has instructed the watchman to make sure there are 32 horses all the time, no more no less. Every circle must have *at least* one horse. He thinks he is smart and so has devised a shortcut to verify this. Regardless of the number of horses in each circle, he just checks the total number of horses along each line of sight.

A traveller comes in the evening and meets the watchman. He has four identical horses and wants to tie them along with these 32 horses. The watchman refuses initially but agrees later when an incentive is offered. The traveller is asked to tie the horses conforming to the '12 in a line' rule. Here is the arrangement. Total = 36 horses.



The story continues. In the middle of the night, the traveller escapes taking his four horses. He also steals four horses. Now only 28 horses are left and the watchman has to arrange them hurriedly, before the king comes and he succeeds! How does he do it? Can you try?

I always thought this a great opportunity to understand the pattern and explore a number of mathematical concepts.

1. Can we solve the problem if the number of horses is other than 32, 36 and 28?
2. Is there a general rule in placing the horses
   a. in the four corners do you place >4 or <4?
   b. Is it possible to solve this problem if the number of horses is odd? How?
   c. What is the theoretical min and max?
   d. Does it work if the shape is a triangle instead of a square?
3. Is this a low floor, high ceiling kind of activity? How?

We shall discuss these later.

But for now, let us revisit the earlier problem of sharing 17 horses among three children. According to the story, a wise man was passing by and volunteered to help. He had one horse with him which he gave them, thus making it 18 horses. He computed 1/2 of 18 (nine), 1/3 of 18 (six) and 1/9 of 18 (two) and added them, i.e., $9+6+2=17$. He took back his horse and proceeded to the next village. The solution may not sound 'mathematically convincing' but everyone said, "Wah, such a smart idea!"

Can we find any pattern and get solutions if the number of horses and the shares of the three children are different? When I challenged the teachers regarding this interesting variation, the response was truly disappointing. They did not see this as an opportunity

to explore the world of puzzles and mathematics. This is sad. In the classroom, we cannot afford to miss ANY opportunity to promote mathematical learning. This is especially true here as the background story is both interesting and motivating.

To illustrate this point, let us explore further with different assumptions. (*Reminder: Add the extra horse that belongs to the traveller before computing*)

Case 1: Two children; they get 1/2 and 1/3. How many horses?
Answer – 5 horses

Case 2: Three children; they get 1/2, 1/3 and 1/8. How many horses?
Answer – 23 horses

Case 3: Three children; they get 1/2, 1/3 and 1/7. How many horses?
Answer – 41 horses

Case 4: Three children; they get 1/2, 1/3 and 1/6. How many horses?
(Hint: Convert this problem into a linear equation)

Case 5: Three children; they get 1/2, 1/4 and 1/8. How many horses?

Case 6: Four children; they get 1/2, 1/4, 1/8 and 1/16. How many horses?

Case 7: Five children; they get ½, ¼, 1/8, 1/16 and 1/32. How many horses?

Did you arrive at a pattern after solving cases 5, 6 and 7? What pattern?

Let us now revisit the King's horse puzzle. Here are some random solutions.

26

6	1	5
1		1
5	1	6

28

5	2	5
2		2
5	2	5

27

6	2	4
1		2
5	1	6

40

1	9	2
8		9
3	8	1

Can you notice any pattern/s here? Can you create more problems similar to this? Is there a possible use of this in real life?

This is a case of number partition, but of a minor-league; not as serious and profound as what professional mathematicians study. Nevertheless, for a school child this is no mean path of joyous discovery. In the hands of persevering and competent children, this can even result in an independent mathematical creation.

Moving on, let us consider this folk puzzle.

There are 100 rotis to be distributed to 100 people. Easy, but there is a catch. This group has some men, some women and some children. Men get 10 rotis each, women get 3 each and children get 1/2 each. The question is how many men, women and children are there?

There are infinite variations of this Diophantine equation[1] that has emerged in almost all parts of the globe and not unsurprisingly, this puzzle is a perennial source of excitement for teachers, children and even parents[2].

A creative mathematics teacher can lead a group of enthusiastic boys and girls on an adventurous mathematical journey and illustrate some elementary ideas relating to variables, equations, trial-and error method, number sense, mapping, problem solving, problem posing, graphs, mathematics in real life, connecting mathematics to other subjects in the school curriculum, etc.

Often, the curious and deep implications arising from folk mathematics are camouflaged by the popular and rustic context in which it finds itself. I am reminded of the essay 'A mathematician's apology' by G H Hardy in which he reflects on chess and mathematics. He says, "A chess problem is genuine mathematics, but it is in some way 'trivial' mathematics. However ingenious and intricate, however original and surprising the moves, there is something essential lacking."

In conclusion, folk mathematics is a wonderful opportunity for a classroom teacher to get children interested in several aspects of mathematics but this requires some special efforts before implementation. It can fortify mathematics teaching and a teacher can use it in the classroom for homework, in assemblies, cultural programs, math mela and so on to offer an enriching and learning experience to the students.

1. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations.

2. Incidentally, the solution is this: There are 5 men, 1 woman and 94 children.

TP

Tangrams in early mathematical education

Lalitha Sundaresan

The author is with the National Institute of Advanced Studies, Bengaluru. She has a Ph.D from the Indian Statistical Institute. She is interested in satellite image analysis and occasionally dabbles with school mathematics. She can be reached at <sundaresan.lalitha@gmail.com>.

Increasingly it is felt that spatial sense is necessary to understand and appreciate mathematics. Several studies have shown that young children exposed to the idea of shapes and how shapes change when rotated, enlarged, etc., are more comfortable with mathematics as they grow older.

In India, formal introduction of geometric shapes and the inter-relationships of shapes is introduced in grade 1. Children are introduced to shapes like triangles, squares, rectangles at this stage. Measurements of size is introduced later on. The idea of big and small are however intuitively taught.

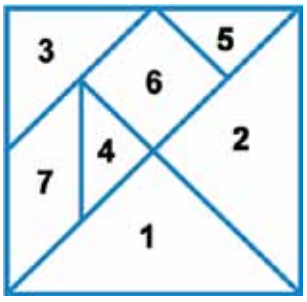
Properties of these shapes like perimeter and area are introduced by grade 3 or 4. Since children are still not familiar with units of measurement and dimensions, these concepts are difficult to introduce and also understand.

Tangram tiles or pieces are very useful in this regard. This article aims to illustrate how tangram tiles can serve the purpose of both fun in the class as well as some learning of these two concepts – perimeter and area.

What is a tangram?

A tangram is a Chinese puzzle consisting of seven flat tiles. The tiles include five triangles, one parallelogram and one square. The original puzzle requires that all seven tiles be used to form a variety of shapes. Although many children are encouraged to work on tangram puzzles to improve their spatial skills, tangram tiles have other uses too.

Making your own tangram tiles is quite simple and requires only a thick square chart paper or cardboard. The following diagram will illustrate how to get the seven pieces.



Start with a square of suitable size and draw the lines indicated in the figure and cut out the pieces.

Teachers will note that the sides of the tiles are not rational numbers. The seven pieces will have the following sides if the original square was of length 1 unit.

Tile	Shape	Side / base	Side / Height	Hypotenuse
1	Right Triangle	√2/2	√2/2	1
2	Right Triangle	√2/2	√2/2	1
3	Right Triangle	1/2	1/2	1/√2
4	Right Triangle	√2/4	√2/4	1/2
5	Right Triangle	√2/4	√2/4	1/2
6	Square	√2/4		
7	Parallelogram	1/2	√2/4	1/4

For the purpose of this article we will follow the labels provided for the tiles.

The perimeter and area of these tiles are also shown in the table below:

Tiles	Perimeter	Area
1, 2	2.14	0.25
3	1.57	0.125
4,5	1.07	0.0.625
6	1.14	0.125
7	1.57	0.125

Develop spatial skills

Early on, children do become familiar with shapes such as triangles, squares, and rectangles. The tangram tiles can be used to emphasize this to start with. It is not necessary to use all the seven tiles to create shapes.

- The teacher can ask children to identify shapes, pick up identical ones in terms of shape, size; ask them to place one tile on top of another to find out if they are identical.
- Demonstrate how two triangles combine to form a square.
- There is no rectangular tile here. Ask children to form a rectangle with the given tiles. Children are bound to come up with several ways to do this.
- Parallelogram is a new shape for them. Ask them to form parallelograms with the given tiles.

Measurements and perimeter

Children can be taught to measure the sides of these tiles using a thread to demonstrate that

i) The sides of the triangles are not all equal.
ii) Two sides of the triangle are equal.
iii) What isosceles triangles are.
iv) Difference between rectangles and squares.

The following two rectangles can be formed by the children.

First ask the children to use a thread to measure the length and breadth of the two rectangles. Are the two rectangles identical? If we remove tiles 1 and 2 do we still have a rectangle?

Now to the perimeter. Take tile 6 which is a square. Starting from one vertex of the square, using a thread, ask the children to go north, then east, south and west to come back to the starting point. Introduce the idea of perimeter at this stage.

The teacher could ask the children to find the perimeters of all the seven tiles. Children may find some surprising results here.

This way of measuring helps to introduce the idea of directions also (north, east, west and south). They could also experiment starting at the south-eastern corner and move the thread north, west, south and east and find that the perimeter or distance travelled is the same irrespective of the route taken!

By grade 4 children are taught to use a foot ruler and they can be encouraged to write down the actual perimeter in centimetres.

At this stage they will also discover that different shapes can have the same perimeter. For example tiles 3 and 7 have the same perimeter, but different shapes.

Area

Area is a more difficult concept to introduce. The teacher can introduce area as the space contained within the perimeter of an object. Larger the space inside, larger the area.

Take for example, a kitchen towel, a bath towel and a rectangular doormat. Spread them on the floor. It

is immediately apparent to the children that the bath towel occupies more space than the door mat and the kitchen towel.

Tiles 1, 2, 3, 4, and 5 are all triangles. The children can be asked to order them in terms of area. Larger to smaller. By placing tile 1 on top of tile 2 they can be taught that they both have the same area. Similarly tiles 4 and 5.

What about tiles 3, 6 and 7?

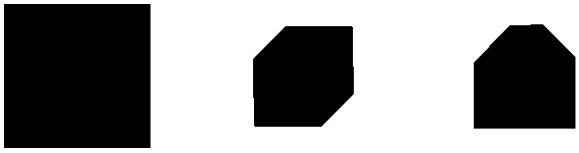
Place tiles 4 and 5 on top of tile 6 to show that the area of tile 6 is the sum of the areas of tiles 4 and 5.

Recall that we have not introduced the notions of measurement units or dimensions.

Now ask the children to find the area of tiles 3 and 7, the parallelogram using the other tiles.

Tangram puzzles

At this stage, introduce the children to the tangram puzzle. The children should use all the seven tiles to create shapes such as the ones given below.



Once these shapes are created using all the seven tiles, tell them that all the three shapes have the same area, because the area is the sum total of the areas of the seven tiles.

Ask the children to find the perimeter of these shapes. Are they all the same? Which one has the largest perimeter?

Introduce the idea that although the area of an irregular shape such as the one given below can be determined by summing up the areas of the regular shapes that constitute it, the same cannot be said for its perimeter.

Another simple exercise will be to find the perimeter and area of the following three shapes using all the seven tiles.

More activities

Children can now be introduced to the idea of how rectangles can be constructed with the same area but different perimeters. Cut out or draw rectangles of size 6 X 6, 9 X 4, and 12 X 3. All of them have the area 36 in appropriate units. Their perimeters are respectively 24, 26 and 30 units.

Let us take another example of two rectangles of sizes 16 X 2 and 9 X 4. Their areas are 32 and 36 respectively. So, children will recognize that rectangle 2 is larger than rectangle 1 in terms of area. Their perimeters however, tell a different story. Rectangle 1 has a perimeter of 36 and rectangle 2 has a perimeter of 26. So if you have to go around these two rectangles, you will cover a larger distance in the case of rectangle 1.

Conclusion

Tangram puzzles help children develop and improve spatial skills. Using the seven tiles of the tangram, children can be given an intuitive idea of perimeter and area of regular and irregular geometric shapes.

These puzzles help to reinforce the point that perimeter and area are not related in any way.

Tangram puzzles can be used to illustrate that irregular shapes having the same area can have different perimeters. In other words shapes having identical areas need not have identical perimeters.

TP

One problem – many solutions

M. Gopalakrishnan

The author became a teacher after working as an engineer for 17 years. He has been in the field of education for 19 years now and is currently teaching math to classes 8, 9 and 10 at The Peepal Grove School, Chittoor, Andhra Pradesh. He can be reached at <truthalonefrees@gmail.com >.

There is no one way to look at a problem. One doesn't gain complete understanding by solving a problem in just one way. A problem can be solved in many different ways. Once a student solves a problem in one way, she tends to move to the next problem and doesn't pause to think whether there is another way to solve the same problem. Trying to solve the same problem in many different ways will deepen one's understanding of the problem. I try to encourage this in my classes.

After explaining the midpoint theorem to class 9 students, I gave them a few simple problems involving triangles to help them understand the concept better. They solved the problems using the midpoint theorem. Then I gave them the following problem and asked them to solve it, assuming that they would solve it by applying the same theorem.

Q1: ABCD is a rectangle. P, Q, R and S are midpoints of the sides in order. Prove that the quadrilateral PQRS is a rhombus. (Fig-1)

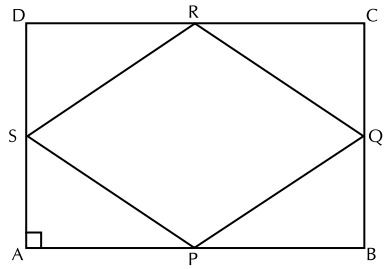
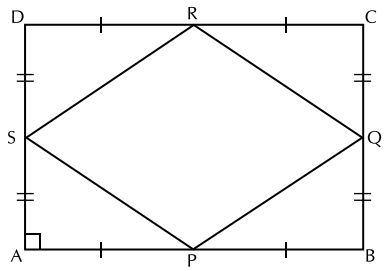


Fig-1

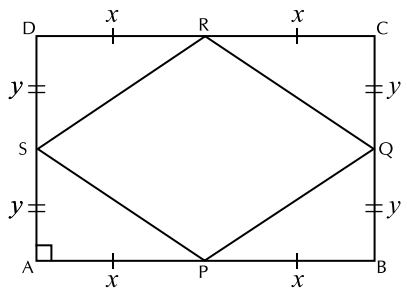
After a few minutes, Vinaya (all names have been changed) approached me and asked, "Bhaiya, is it necessary that we use only the midpoint theorem for solving this problem?" Though I wanted them to use the midpoint theorem in order to practice it, I told her "No. There is no such requirement. You can use any other concept you know already. Only make sure that your reasoning is correct." She took some time, identified the four congruent triangles using the side-angle-side axiom and solved it as shown in Fig-2.



$\Delta SAP \cong \Delta QBP \cong \Delta QCR \cong SDR$ (side – angle – side congruency axiom)
$\therefore$ PQ = QR = RS = SP (corresponding parts of congruent triangles)
$\therefore$ PQRS is a rhombus

Fig-2

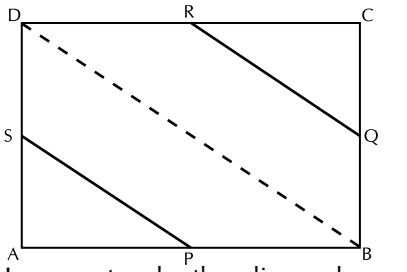
I was impressed. I asked the class whether anyone else could think of a different solution. Arpit saw the four right triangles and hit upon the idea of using the Pythagoras theorem. (Fig-3) I was even more impressed. I appreciated both of them and shared their solutions with the entire class.



AP = PB = CR = RD = x
AS = BQ = CQ = DS = y
By Pythagoras Theorem,
$PQ = QR = RS = SP = \sqrt{x^2 + y^2}$
∴ PQRS is a rhombus

Fig-3

Later I asked them whether they could solve the same problem using the midpoint theorem. Students spent some time and figured out that by drawing a diagonal, the rectangle could be divided into two triangles. (Fig-4)



In a rectangle the diagonals are equal
∴ BD = AC = x
$PQ = \frac{1}{2} AC = \frac{x}{2}$ *(By Midpoint Theorem)*
similarly,

$RS = \frac{1}{2} AC = \frac{x}{2}$

$PS = \frac{1}{2} BD = \frac{x}{2}$

$QR = \frac{1}{2} BD = \frac{x}{2}$

∴ PQ = QR = RS = SP
∴ ∴ PQRS is a rhombus

Fig-4

This made me realize how creative students could get in coming up with different solutions to the same problem. On another occasion, I gave them the following problem:

Q2: ABCD is a parallelogram. P and S are midpoints of AB and AD respectively.

Prove that area APS = $\frac{1}{8}$ area ABCD (Fig-5)

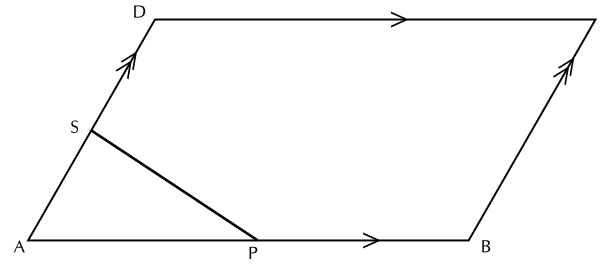


Fig-5

You may try to solve and see how many different solutions you can come up with. My students came up with four different solutions.

For the following problem they came up with three different solutions:

Q3: Triangle ABC is right angled at B. P is the midpoint of AC.
Prove that PA = PB = PC (Fig-6)

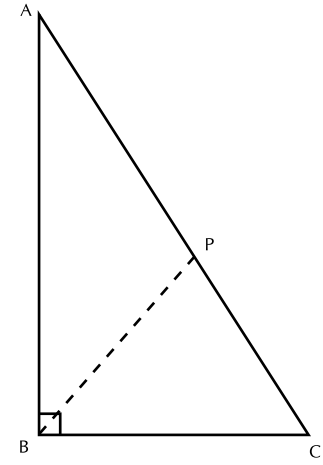


Fig-6

Conclusion

While teaching any concept, students should be asked to solve a problem in as many different ways as possible, including the application of the recent concept taught. This motivates them a lot, makes them look at the same problem from different perspectives, strengthens their understanding and helps them be creative.

The art of mathematics

Saroja Sundararajan

The author, who was the first CBSE teacher to get the Best Teachers National Award in 1986, has had a very successful teaching and teacher training career spanning over five decades. She was the Director of the Teachers' Centre at Springdales School, New Delhi and is the author of several textbooks. She now works on a freelance basis from Chennai. She can be reached at <saroja.tvs@gmail.com>.

Usually, it is a small percentage of students who really look forward to a maths class. For the rest, it spells gloom. Why? What should we do to help rid our students of their math phobia?

Imagine a plain brown cake placed on a table. Next to it is another cake with colourful toppings, laced with mouth-watering icing and beautiful decoration. Which do you think a child will choose? Almost certainly, the second one. Why? Because the presentation is very attractive.

In junior classes when you are teaching, say 'subtraction', you can make it attractive for your students. Draw a basket full of apples on the board and ask your students to think of reasons why apples are disappearing from the basket. As your students give their reasons, they cross out one apple in the basket.

Your students can be creative and funny with their reasons. Here are some of the reasons I have been given by my students:

"A teeny-wheeny mouse came and stole one apple at night."

"An alien from Jupiter gobbled up one apple."

As your students give their reasons, write 10 apples – 1 apple = 9 apples.

9 apples – 1 apple = 8 apples and so on.

Such exercises will not only help evince an interest in the subject for the students, but will also enhance their creativity. Permanent learning takes place when you integrate art with mathematics!

In this article I will show you how you can include both the visual arts (drawing, colouring, cutting, etc.) and the performing arts (dance, music, storytelling, etc.) in your math classes.

Let us see examples of each and what effect they have on the learning skills of the student.

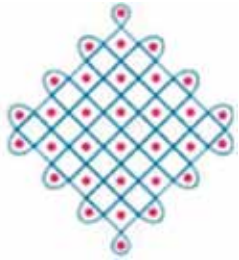
Visual arts

a. Intuitively learn a higher level concept.

Task 1: (class 6 – number patterns)

Draw a *kolam* (rangoli pattern) between the two existing kolams by decoding the pattern.

1,3,3,1



1,3,5,7,7,5,3,1

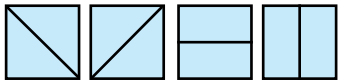
Mathematics combined with art not only brings out the number sense but also creativity in the student. A student going to higher classes learns that the sum of the first n consecutive odd integers

will yield a square number 'n^2'. How can we lead the students to grasp this concept intuitively in the lower classes? Look at the odd integers placed in rows, together they form a beautiful square each. Look at each of them sideways!

b. Promote lateral thinking (classes 7, 8 – congruency)

Task 2:

- Fold and cut out the BIGGEST SQUARE from a rectangular sheet of paper.
- Draw different lines inside the square to show that the lines divide the square into two halves.
- Ask your students to think out of the box, think of solutions other than these obvious ones.



c. Encourage perspectives: (class 8 – three dimensional solids)

Task 3:

Ask students to draw a square as above. Tell them that they should turn this figure into a 3D figure. It should look as though we are entering a room through a long corridor. How will you modify the original square to get such an illusion?

This is what *perspective art* is all about.

Perspective is an art technique for creating an illusion of three-dimensions (depth and space) on a two-dimensional (flat) surface. Perspective is what makes a painting seem to have form, distance, and look "real."

The mind is cheated into seeing something that does not exist. It requires skill to produce such art.

d. Correlate numbers with nature: (class 8 – real numbers)

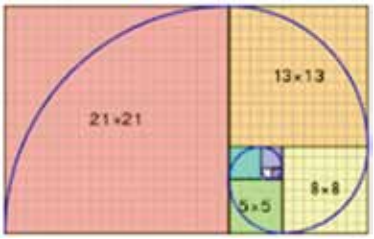
Fibonacci was an Italian mathematician from the Republic of Pisa, considered to be "the most talented Western mathematician of the Middle Ages".

He discovered a number series known as the Fibonacci series.
1,1,2,3,5,8,13,21,34.....

The sum of any two consecutive numbers in the above series, gives the next number in the series: $1+1 = 2$; $1+2 = 3$; $2+3 = 5$; $3+5 = 8$ and so on.

Task 4:

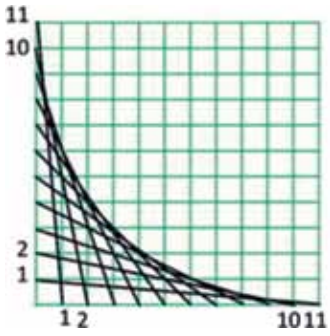
When each of the numbers is depicted by a square, drawn adjacent to the previous one and a curve is drawn as given here, it would form the Fibonacci spiral. This spiral is found abundantly in nature.

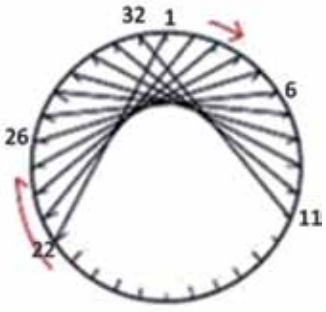


e. Design jewellery by creating curves with straight lines: (class 8 – circles)

A seemingly impossible task made possible by juggling numbers. Experimenting by joining numbers differently will produce different designs.

Task 5: Using these ideas, create jewellery designs.





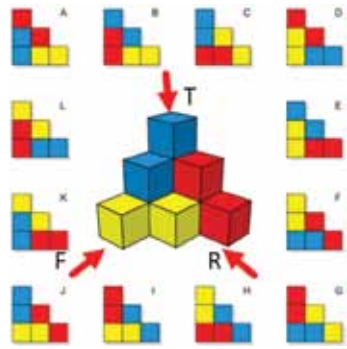
f. Enhance the skill of recognizing and sifting which will help in building architectural skills: (class 8 – 3D shapes)

In perspective drawing, students learn to convert a 2D visual into a 3D

visual. Here, conversely, they will learn to represent a 3D figure in its 2D form. This skill comes in handy when students are working as architects and have to draw an elevation or a plan of a building. The elevation is the front or side view which is the vertical visualization of an object. The top view is the horizontal view of the object.

Task 6:

Twelve 2D forms of the central figure is given here. The centre piece is the 3D form.



Is it possible that you can identify the correct 2D form which matches the central 3D figure?.

Performing arts

1. ***Music:*** Music and mathematics go hand in hand. Music is based on mathematics in its metre, rhythm, pitch, tone, etc.

   Pythagoras, the famous mathematician, said that the string of an instrument determines its pitch. In the primary classes teaching mathematics through song and dance is the best form of teaching.

https://www.youtube.com/watch?v=lqw2iAv3Yb4
https://www.youtube.com/watch?v=ZzeDWFhYv3E
https://www.songsforteaching.com/math/multiplication/multiplyingmagic.php

These websites will give the teacher ideas as to how music can be used in the math classroom.

2. **Storytelling:**

a. Spatial skills and fractions through narration (std 6, 7, 8 – geometrical shapes)
   Tangram is an ancient Chinese puzzle. A square figure is divided into seven pieces as follows:
   1) 2 big triangles
   2) 1 medium sized triangle
   3) 2 small tringles
   4) 1 square
   5) 1 parallelogram

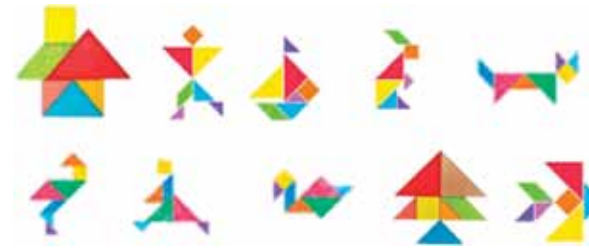
   The Chinese used to spend their free time making meaningful figures with all seven pieces assembled together.

Here is the tangram.

Photocopy this square and keep at least 10 copies of it.

Each copy of the square should be neatly cut into the seven geometrical pieces and placed in separate envelopes.
Given below are several figures made using the tangram pieces.
Note that all the seven pieces are used in each figure.
Make up a story using these figures.
According to the sequence of your story, on a chart paper, assemble and paste the figures.
Read out your story to the class.

Here are the figures that you will use for your story:

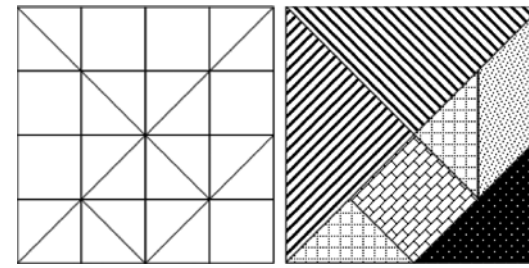


Storytelling helps in

- Bringing out creativity in the student.
- Developing the art of narrative skills.
- Shedding inhibitions and
- Developing spatial skills in mathematics while cutting, assembling and pasting the pieces.

Surely by now you can appreciate the immense value of integrating art into the mathematics curriculum. But remember these activities have to be woven into regular classroom teaching rather than be given as extra project work, only then will the objective be achieved.

A sample lesson plan integrating art with mathematics



- On a graph sheet of paper, draw the tangram square.
- Make multiple copies.
- Cut them out, colour or shade each piece differently and keep them ready for the activity (as shown in the black and white image on the previous page).

If the whole square is 1 unit square, calculate the area of each piece as a fraction of the whole. For example, each big triangle = ¼ of the whole.
Similarly calculate for each piece.

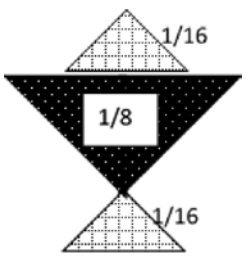
Learning objectives:
At the end of the lesson, the student will be able to:
- Draw a tangram square, dividing it into the seven geometrical shapes.
- Colour each piece aesthetically.
- Cut out each piece accurately.
- Calculate the fraction that each represents of the whole and write the fraction on each piece.
- Choose and assemble the pieces in such a manner that the total of the fractions equals the fraction asked to be formed.
- Add fractions with ease.

Materials required: A graph sheet, pencil, ruler, colour pencils, a pair of scissors, glue.

Lesson task:
Make multiple copies of the pieces.

Task 1:
- Choose the appropriate shapes that you require so that they add up to the fraction asked for.
- Assemble them to make a meaningful figure.
- In your notebook, either paste them or copy and draw the shape.
- Repeat the task for each of the following fractions.
  a) 1/4 b) 1/2 c) 3/4 d) 1/8 e) 5/8 f) 1

An example has been given for you.



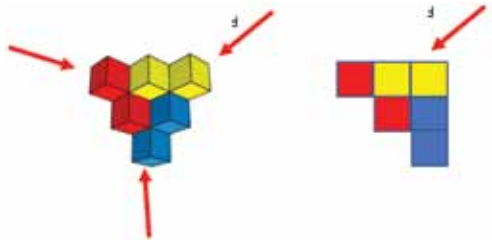
1/8 + 1/16 + 1/16 = 1/4
Ice-cream cup

Task 2:
To the above figure, add one more piece and make a different shape so that the total fraction of the new shape is ½.

Think...
1. Were you able to complete the tasks with ease?
2. Which part of the activity did you find difficult? Why?
3. Can you make your own activities using tangrams?

(This lesson plan can be used when students are taught fractions, geometrical shapes or polygons.)

TP



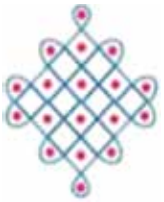
Task 6:

Task 5:

Task 3:

Task 2:
These are also halves of each whole.

Solutions:
Task 1:



1,3,5,5,3,1

The magic of magic squares

T R Mani

The author (born in 1934) is a mechanical engineer who retired in 1992 after a career with MICO Bengaluru. After retirement he developed an interest in mathematics and created many magic squares with new concepts. He has conducted math exhibitions and workshops for training students and teachers. For his work in mathematics, the Ramanujam Museum and Math Education Centre honored him in 2001 & 2019. He has a Facebook page Mathemagic Squares. He is available on WhatsApp, 9444901330. He can also be reached at <sheelnand25@gmail.com>.

Constructing magic squares has been a pastime of many eminent mathematicians. Srinivasa Ramanujan, in addition to a number of astonishing mathematical formulae, was interested in entertainment mathematics as well. The first chapter of his two volume book is all about magic squares. In this article we will see a few examples of magic squares.

Magic squares can be both odd and even, depending on the total number of squares. An odd magic square will have odd rows and columns and even magic square will have even number of rows and columns. In any magic square, the sum of all the rows, columns and the two diagonals will be the same. This is called the magic sum of that square.

The algorithm for constructing an odd magic square is simpler than that for an even magic square. In general, the same algorithm works for an odd magic square of any size. This is not so in the case of an even magic square.

Odd magic squares

The simplest magic square is a 3X3 square. The transformation is indicated below.

1	2	3
4	5	6
7	8	9

→

2	7	6
9	5	1
4	3	8

The algorithm can be summarized as – corner numbers move three boxes forward. Middle numbers move one box backwards. This method is applicable for any set of natural numbers, odd or even or any block of nine numbers from a calendar.

This magic square also has many super hidden identities

$2^3+7^3+6^3+(2+7)^3+(7+6)^3+(6+2)^3 =$
$4^3+3^3+8^3+(4+3)^3+(3+8)^3+(8+4)^3$

Novel method for 5X5 magic square

Diagonal starting with 1 to form the middle column. Maintain row order

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

→

4	5	1	2	3
10	6	7	8	9
11	12	13	14	15
17	18	19	20	16
23	24	25	21	22

Middle row to form the diagonal starting with 11. Maintain column order.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Magic sum = 65. Method applicable to any odd magic square

Even magic squares

Even magic squares are more complex to construct and have many variations and levels of complexity. Let us see some examples.

Some of these can be constructed using the arithmetic series of numbers. They can also be constructed using a random selection of numbers.

Sri Rama Chakra

Sri Rama Chakra, found on the last page of certain versions of the Panchangam, is a magic square marvel. It is also called the pandiagonal magic square.

9	16	5	4
7	2	11	14
12	13	8	1
6	3	10	15

In a conventional magic square, the numbers in all the columns, rows and diagonals add up to the same total. In this magic square, 42 other formations of four numbers also add up to this total. Some examples are $9+4+15+6$, $9+16+2+7$, $9+5+8+12$, $7+16+10+1$, $9+16+6+3$ and $16+5+10+3$. There are more than 100 other interesting identities. This is the reason it is called a diabolic magic square as well.

Constructing an even magic square

Even magic squares can be constructed using two important identities, which I will illustrate using the Sri Rama Chakra as an example.

1. The sum of the middle two numbers of any extreme (top, bottom, left or right) row/ column should be equal to the sum of the corner numbers at the other end.
   In the above case 16 + 5 = 6 + 15 or 7 + 12 = 4 + 15
2. The sum of the numbers at the ends of any diagonal should be equal to the sum of the middle two numbers of the other diagonal.
   In the above case 4 + 6 = 2 + 8 and 9 + 15 = 13 + 11

Since there are many ways of choosing numbers to satisfy an identity, many variations of a magic square with the same magic sum are possible. However, it takes a lot of effort and many trials to get a magic square without repetition of numbers.

Date magic square

Topping the innovations in magic squares is the use of random numbers as against the use of series numbers. In this magic square the top row contains a date in the dd/mm/yy/yy format. This automatically fixes the magic sum which is dd + mm + yy + yy.

It uses the two interesting identities already mentioned. The algorithm for making date magic squares can be learnt by any middle school student.

Here is a magic square where the first row represents Ramanujan's date of birth – 22nd December 1887.

22	12	18	87
31	93	07	08
60	19	20	40
26	15	94	04

The magic sum is 22 + 12 + 18 + 87 = 139. The four numbers in the top row are given. By repeatedly applying the two identities already mentioned, we can fill three numbers in all the rows and columns.

$12+18=30=26+04$, $22+04=26=19+07$,
$26+87=113=93+20$ & $22+26=48=08+40$

Other empty cells are filled by adding the three numbers and subtracting from the magic sum.

Easiest magic square

One of the easiest algorithms for constructing an even magic square is given below.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

→

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

The method is simple – reverse both the diagonals and retain the others.

This square also contains other interesting identities
$16^2+2^2+3^2+13^2 = 4^2+14^2+15^2+1^2$
$(16+2)^2+(2+3)^2+(3+13)^2+(13+16)^2 =$
$(4+14)^2+(14+15)^2+(15+1)^2+(1+4)^2$
$16X2+2X3+3X13+13X16 = 4X14+14X15+15X1+1X4$

1-2-3 method for mega magic square – example 8X8

				RAJA	RANI				
	1	63	3	61	60	6	58	8	
	56	10	54	12	13	51	15	49	
	17	47	19	45	44	22	42	24	
LOC	40	26	38	28	29	35	31	33	LOC
	32	34	30	36	37	27	39	25	
	41	23	43	21	20	46	18	48	
	16	50	14	52	53	11	55	9	
	57	7	59	5	4	62	2	64	

Starting with 1, alternate ascending numbers in alternate box. If RAJA comes, RANI with the next number. If NOT skip.

At LOC (Line of Condition) next number comes. Thereafter ascending to descending. Starting with 63 in the second box, alternate descending numbers in alternate box. If RAJA comes, RANI with next number. If NOT skip. At LOC, the next higher number. Thereafter descending to ascending.

Magic sum: $\frac{n(n+1)}{2} x \frac{1}{8} = \frac{64x65}{2} x \frac{1}{8} = 260$

The method is applicable to magic squares in multiples of four and also suitable for a 100 X 100 mega magic square.

Magic product squares

Learning simple magic squares, R Adithya, a student from my math enrichment workshops, thought out-of-the-box and created a magic square using products instead of sums!

15	04	20	02
20	02	03	20
01	50	08	06
08	06	05	10

Magic product – 2400
First row contains date of birth
Repeated (underlined) numbers multiplied by 3.5

15	04	70	02
20	07	03	20
01	50	08	21
28	06	05	10

Magic product – 8400
Further repeated (underlined) numbers multiplied by 2

15	08	70	02
20	07	03	40
01	50	16	21
56	06	05	10

Magic product – 16800
Final without any repetition

Magic squares offer a high degree of creativity apart from teaching arithmetic, instilling interest in numbers and a liking for mathematics, making it an easy subject to assimilate.

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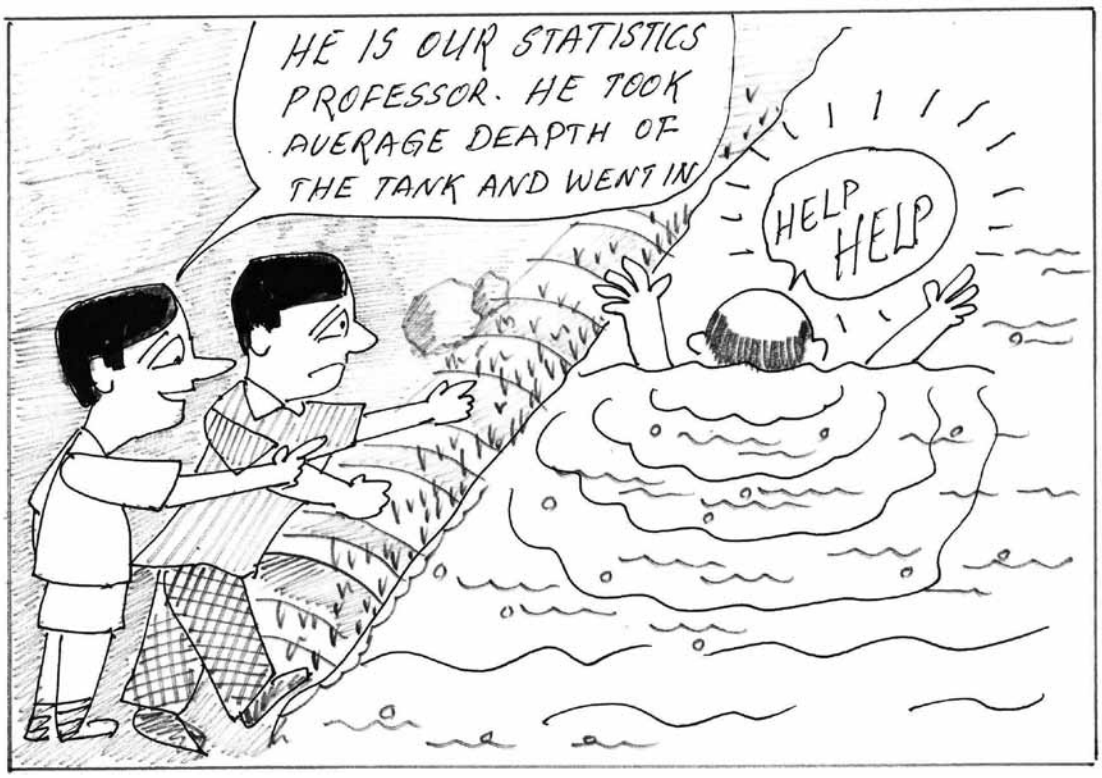


Making smart use of technology

Sinny Mole

The author has been a math teacher and Assistant Coordinator, Primary-2 at Kokilaben Dhirubhai Ambani Vidyamandir, Reliance Greens, Jamnagar. She can be reached at <sinnymolepm@yahoo.in>.

In this fast developing world, practically every sphere of life has experienced a technological revolution. Digital technologies are changing the way we communicate with our families and friends and have impacted the way we are learning. These days it is very common to see private and government school teachers and students using technology.

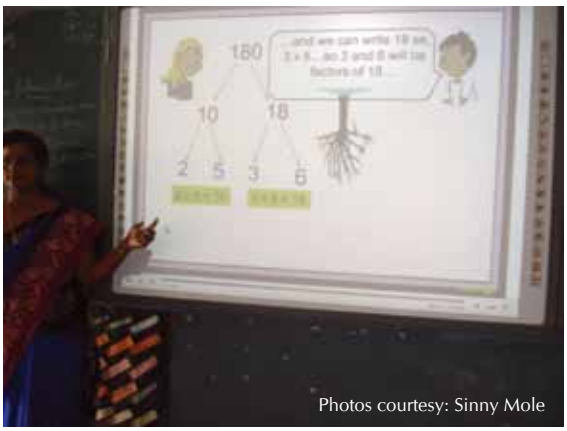
The process of teaching-learning can be enhanced greatly by using technological tools. The smart board is an example. 'Smart Classroom' equipment is a solution designed to help teachers overcome new challenges and develop students' abilities and performance.

The last decade has seen a dramatic change in the way we teach in schools. The school where I used to teach decided (in my last few years there) to adopt modern teaching aids in the form of interactive smart boards and projectors. Until then, an interactive smart board in every classroom was considered a luxury. When I look back, it still seems unbelievable – one day when we walked into our classrooms, we saw the smart boards waiting for us! An "aha" moment for all of us.

The use of smart boards can help make mathematics teaching easier for teachers and understanding of concepts simpler for students. The smart classrooms are networked to a common server where all the digital content is available. Teachers can access the lessons they want to teach and can use appropriate multimedia content and information to make the learning more attractive and effective.

Smart boards make learning fun for students. It can turn a boring lecture into an engaging and interactive session. Instead of just talking about the topic, digital boards engage the learner by displaying content in the form of animation and visuals. It is true that when we learn through visuals, we grasp the subject more easily.

We had started implementing activity-based teaching in primary mathematics. Adopting modern teaching facilities made it much more convenient to implement this programme. For example, while teaching the area-perimeter lesson, square grids, paper cut-outs, geoboard, and string activities were used for demonstrations. With the smart board we were able to provide more hands-on activities and do quick revisions as well.



Photos courtesy: Sinny Mole

Resources have an important role to play in allowing teachers to model or demonstrate mathematical ideas and in supporting the development of mathematical understanding and thinking in children. The good thing is that a number of different teaching aids and accessories exist that help to make the whole process of math learning interesting. While some of the resources can be obtained only upon subscription, others are freely available on the internet. To meet the classroom specific requirement, one can also adapt or develop materials in-house on the computer.

For example, digital content can be embedded with hyperlinks to display graphs and videos within the body of materials to aid understanding and to demonstrate a concept. Such tools will help the teachers instantly illustrate concepts during a lesson. With just one click, teachers can bring the outside world into their classroom. For example, pictures of a tiled floor, playground, fenced garden and walkways can be used to explain area and perimeter. This helps save a lot of time and the students get a feel of the points discussed in the class.

Math websites for teachers

Teachers should learn the skills required to successfully plan and teach motivating lessons. It is essential for a new teacher to get ready to spend adequate time for developing interesting lessons. Smart classroom service providers organize special training and hands-on sessions for teachers to enhance their capability of handling the new tech devices.

To strengthen the lessons and classroom teaching, teachers can utilize math websites, knowledge databases, online audios and videos and presentations. The problem is finding the best ones. Many websites have games, interactive activities and resources such as worksheets – they vary in quality and usability. A close study and preview of the material is very essential to ascertain its usefulness.

Here are some useful math websites:

- https://www.youtube.com/channel/UCEg3PHVR0vt3azbs_Qk_6cw – free education platform
- https://modernchalkboard.com/ – smart board notebook lessons
- https://www.youtube.com/user/numberphile/featured – math videos on numbers
- https://www.youtube.com/user/tecmath – mental math tricks
- https://www.youtube.com/user/mathantics/featured – video tutorials on everything related to Math
- https://www.youtube.com/channel/UCtBtcQJ8_jsrjPzb8i1tOsA – a mathematics education
- https://www.youtube.com/user/HEGARTYMATHS/featured – math channel
- https://www.youtube.com/user/khanacademy/playlists – math courses and video tutorials
- http://www.topmarks.co.uk/Interactive.aspx?cat=1 – math games
- http://www.coolmath4kids.com/ – math games
- http://www.shodor.org/interactivate/activities/
- http://illuminations.nctm.org/
- http://www.teachingideas.co.uk/ – teaching resources
- http://www.shodor.org/interactivate/activities/
- http://www.aaamath.com/ – interactive arithmetic lessons

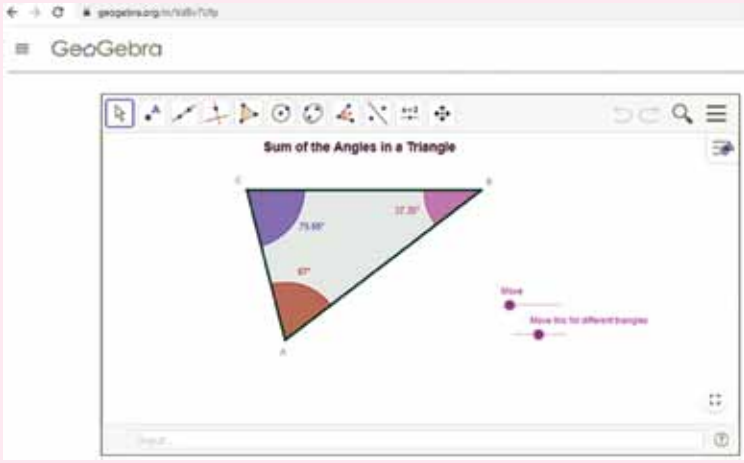
Write a decimal for each of the following:
Express 72 as a product of prime factors.
Step 7
Now express 72 as a product of prime factors.

GeoGebra – unleash your potential

GeoGebra is a dynamic and powerful teaching tool for math teachers. This software is suitable for all levels of education from primary to university and brings together geometry, algebra, spreadsheets, graphing, statistics and calculus in one easy-to-use package.

There are several ways to use GeoGebra for teaching and learning. GeoGebra allows teachers to create an interactive learning environment with multiple representations, computational utilities, documentation tools, and web-friendly features. It promotes experimental and discovery learning for students while visually interacting with geometry, algebra, graphing, etc. This software supports constructions with points, lines, circles, ellipses and parabolas, etc.

For example: To show that the sum of the internal angles of any triangle is 180 degrees, you can drag the vertex of a triangle so as to change its size and shape. See what kind of angle combinations you can come up with. The sum of the internal angles will always be 180!



Teachers can download GeoGebra software (https://www.geogebra.org) and practice it in their classrooms.

Teachers use the internet and gather a lot of study materials appropriate to their area of teaching and learning. These resources collected by individual teachers in school are put together and are referred to as the digital repository. Digital content of all kinds are stored here. The repository supports mechanisms to import and export materials and teachers can locate and retrieve useful digital resources.

These resources bring variety to teaching and learning experiences and act as visual aids to motivate children to build mental images to connect 'classroom mathematics' with application in the real world. This highlights the need for teachers to be clear as to why they provide specific resources to recognize and establish the links between practical task, visual imagery created and selected aspects of the lesson planned.

There exists a lack of understanding on the part of some teachers as to how to represent mathematical concepts. However, it is important that you don't use these resources to simply add 'fun' to your lessons.

Teachers should have a rich understanding of the lesson they are teaching and appreciate how knowledge in their subject is created, linked and applied to classroom situations. The most effective teachers have the ability to use a wide range of resources – concrete objects, images, language and symbols – to represent mathematical concepts for their students.

A teacher's skill can be developed through knowledge sharing sessions and peer coaching.

No doubt, teaching is a complex activity. It is therefore important for teachers to know what materials to use, when to use them and how, in order to make learning and teaching powerful. Used appropriately, digital learning resources can add significant value to enhance the quality of teaching.

Schools must promote the effective use of digital technologies safely and responsibly to prepare students for the demands of an ever-changing world.

TP

Math in action

Suma Vivekanandan

The author is a mathematics teacher and primary coordinator in Atul Vidyalaya, Atul, Gujarat. She can be reached at <suma_vivekanandan@atulvidyalaya.ac.in>.

I have noticed that students often get confused between AREA and PERIMETER. They also make mistakes in writing the correct unit. The activity described here helped clear these two concepts for my students. This activity also helps students understand why we need a standard unit to measure and what the standard unit for area is.

Divide the class into groups of 4 to 5 students. Instruct each group to fill the inside of a tile on the floor or the table top using different objects – notebooks, newspaper, crayons or bottle caps.

After they finish ask each group how many (number) objects they used to fill up the inside of the tile or table top.

Each group will give you different answers like 20 notebooks or 10 newspapers or 100 bottle caps, etc.

Have a class discussion. Ask your students why despite the size of the tile/table top being the same, each group had a different answer.

Photos courtesy: Suma Vivekanandan

Here you can discuss the need for a standard unit.

After this activity and class discussion, students themselves will conclude that Unit blocks can be used to measure the AREA and the unit of area is square unit.

Here you can ask them why it makes sense to use square units. Allow them to think and give answers. You may get different answers. After listening to their views, tell them that any standard must easily be "constructed" on the spot and it is not convenient to transport a standard from place to place. From this point of view a square is the easiest figure to specify as it has just the one dimension, its side! The square can also fill (tessellated) up any area without leaving any gaps!

You can also incorporate another activity to show them the difference between area and perimeter.

- Make several square cards (say the size of a chessboard square). Take the side as 1 unit and area as 1 square unit.
- Suppose you give the students 12 cards and ask them to make rectangles of different sizes. You can make a 2 X 6, a 3 X 4 or a 1 X 12 rectangle.
- Ask students – what property all these rectangles have in common (area!)
- Ask them to find the perimeter of each of these figures. They will have different perimeters. Find the one with the least perimeter and the one with the most.
- This shows that same areas can have different perimeters.
- Have classroom discussion:- If you think of these shapes as the surfaces of dining tables then
  - o The number of plates and dishes you can place on the table will depend on the area.

- The number of people who can sit around the table will depend on the perimeter!

- Ask the students to use a thread of a given length and create different figures which will have the same perimeter but different areas.

What time is it? Being able to tell the time is an important skill that children learn in primary school. This activity described not only helps children learn to tell the time but also to convert the 12 hour format into a 24 hour one.

Start your class by asking the students why a clock displays numbers only from 1 to 12. Allow your students to brainstorm. Ask them to come up with common proverbs related to time, such as "Time and tide wait for none," or "A stitch in time saves nine" to get them thinking about the subject.

The following activity enables students to read and write time in different ways.

This activity can be done inside or outside the classroom.

Ask 12 students to sit in a circle (like a clock) holding placards. Each card will have three numbers on it. Right on top will be time in the 24 hour format followed by time in the 12 hour format and finally time in minutes at the bottom.

- Place a shoebox (or any empty box) with time chits inside.

13 1 5 minutes	14 2 10 minutes

- The chits should have time written in all possible ways. 11:55, 5 minutes to 12, 55 minutes past 11, 5 minutes past 2, 2:05, half past 6, 18:30, etc.
- Children take turns, walk to the centre of the circle, pick up a chit from the box, read out the time and show it on the clock (that their classmates have formed) using their hands or two sticks.

These kind of activities also highlight the role of language in mathematics.

11.55 is eleven fifty five or fifty five minutes past eleven or five minutes to twelve. This is an opportunity for the teacher to talk about what the day means to the common man and to a scientist/mathematician. For the common man a day is 12 hours (the other 12 being night). So we have am and pm. For the scientist/mathematician a day is 24 hours (it is a cycle). There is no am or pm. The starting point is 12 midnight.

Learning to read time in the traditional way is important even in this digital age for it improves a child's motor and cognitive skills besides his mathematical skills. Understanding time the traditional way is also necessary to later understand concepts like time zones and international time.

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